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The target space geometry of $N = (2, 1)$ string theory

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Abstract

We describe the $\mathcal{O}(\alpha'^0)$ constraints on the target space geometry of the $N = (2, 1)$ heterotic superstring due to the left-moving $N = 1$ supersymmetry and $U(1)$ currents. In the fermionic description of the internal sector supersymmetry is realized quantum mechanically, so that both tree-level and one-loop effects contribute to the order $\mathcal{O}(\alpha'^0)$ constraints. We also discuss the physical interpretation of the resulting target space geometry in terms of configurations of a $2 + 2$-dimensional object propagating in a $10+2$-dimensional spacetime with a null isometry, which has recently been suggested as a unified description of string and M theory.

1 Introduction

The suspicion that the fundamental, short distance description of string theory may not involve strings has existed for a number of years, based on the theory's behavior at high energies and at large orders in perturbation theory (for a review of these arguments with references, see the final two chapters of Polchinski (1994)). In addition, developments in string duality (For a recent review and references see Hull (1995)) have provided a picture of the vacuum structure of string theory which casts some doubts on its status as a fundamental theory. At various extremes of the moduli space of vacua a given string theory may have a weakly coupled description as a different string theory, as 11-dimensional supergravity, as "M-theory" (Schwarz 1995b,c), or as some other p-brane theory (note however that Hull (1995) has suggested that the perturbative states of

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the supersymmetric effective field theories are always strings or 0-branes). Furthermore, p-branes wrapped around homology cycles of the compactification manifold sometimes turn into string states when one condenses them, as happens in conifold transitions (Greene, Morrison and Strominger 1995). Wrapped p-brane states may also be related by duality to string states (for a recent review and discussion see Townsend 1995). This has led some to suggest there may be a more "democratic" theory in which strings are treated on the same footing as other p-brane excitations (Becker, Becker and Strominger 1995; Townsend 1995).

Still, there is little indication as to what the underlying theory might be. M-theory has no definition beyond its being an 11-dimensional theory with supersymmetric 2-branes; the quantum theory of 11-dimensional supergravity is unknown; and the analog of the Polyakov action for p-branes with $p > 1$ is nonrenormalizable, partly due to the coupling to worldvolume gravity (for a discussion of this point see van Nieuwenhuizen 1987). Furthermore, one could argue that these dualities are merely features of the low energy theory, and that one particular object defines the short distance theory. As an analogy, one may write a theory of strongly coupled electrons in a one-dimensional metal, described by the massive Thirring model. There will be a weakly coupled description in terms of the Sine-Gordon model (Coleman 1975), but the theory defined at the lattice scale really is a theory of electrons. So in the case of "string" theory there seems to be no reason to privilege any one p-brane state as fundamental, but there is no obvious reason not to, beyond the high energy and large-genus problems alluded to above.

One possibility for a more fundamental description of string theory is suggested by some recurring hints that the worldsheet might be secretly four dimensional, propagating in a spacetime with more than one timelike signature. Atick and Witten (1988) suggested that the time direction might be complexified, based on an attempt to interpret the high-temperature behavior of string theory. Witten (1988) suggested that both the worldsheet and spacetime might be secretly complexified. This was based on a comparison of instantons in an orbifolded topological σ-model with classical configurations dominating high energy string scattering processes (Gross and Mende 1987,1988;Polchinski 1988). Blencowe and Duff (1988) conjectured that the maximum possible spacetime dimensions for a p-brane theory were 12; this arose from demanding equal Bose and Fermi degrees of freedom on the worldvolume, while allowing for a spacetime supersymmetry group other than the super-Poincaré group. As an example they mentioned a $2+2$ dimensional object in a $10+2$ dimensional spacetime that might be related to the Type IIB string by double dimensional reduction. More recently, Hull (1995) has suggested an $11+1$-dimensional "Y-theory" in order to explain certain 6-dimensional supergravity theories with exotic soliton spectra.

Still more recently, Kutasov and Martinec (1996) have found that the bosonic string in 26 dimensions; the heterotic and type IIB strings in 10 dimensions; a

bosonic 27 dimensions; and a supermembrane in 11 dimensions can all be found as different vacua of the $N = (2, 1)$ supersymmetric heterotic string constructed by Ooguri and Vafa (1991b). The open string and examples involving orbifold and orientifold compactifications of string- and M-theory have been found found by Kutasov, Martinec and O'Loughlin (1996). In these constructions the target space is in fact a 2+1-dimensional object, with a 2+2-dimensional worldvolume, living in a flat $10 + 2$-dimensional (or $26 + 2$-dimensional if we give up target space supersymmetry); one or two of these dimensions are gauged away by a left-moving timelike or null $U(1)$ current on the worldsheet of the $N = (2, 1)$ string. Which string or membrane theory one gets depends on how one realizes the left-moving $U(1)$ and how one performs the GSO projection. The target space theory has a lot of structure: it seems to be self-dual gravity coupled to self-dual Yang-Mills fields, in the presence of a covariantly conserved timelike or null Killing vector (Ooguri and Vafa 1991b; see also Pierce 1996). This might be enough structure to allow the $2+1$-dimensional object to be quantized directly.

Simultaneously, Vafa (1996) has suggested a $10 + 2$-dimensional "F-theory" in order to explain the $SL(2, \mathcal{Z})$ duality of type IIB string theory (Schwarz 1995a) directly in 10 dimensions; he also suggested that this was required by gauge fields living on the worldsheets of the D-strings of IIB string theory. The $SL(2, \mathcal{Z})$ duality suggests a hidden torus whose modular group is precisely this duality group, and the gauge field on the D-string requires an increase of 2 in the critical dimension due to the $U(1)$ worldsheet ghosts. Vafa further suggested that a geometrization of the $U(1)$ algebra would add an additional two dimensions to the worldsheet, though how this comes about and how these extra two dimensions are mapped into spacetime is unknown. Tseytlin (1996) also simultaneously proposed a $11 + 1$ dimensional theory of Dirichlet 3-branes (with a worldvolume signature of $(3, 1)$), based on an interpretation of the gauge fields living on the self-dual 3-brane of type IIB string theory.

Given this small explosion of suggestions of a theory of 4-dimensional worldvolumes in 12 dimensions, we would like to begin to understand what precisely this theory might be. The $N = (2, 1)$ string gives us a solid starting point; it has all known string theories and M-theory as different vacua, and it should have a definite target space theory which would describe the dynamics (whatever this means in the presence of 2 timelike directions) of these 3-dimensional objects. However, computations from the point of view of the $(2, 1)$ string worldsheet seem rather unwieldy. The vertex operators of the underlying string theory describe at best linearized fluctuations of the target worldvolume; and describing the quantum mechanics of the worldvolume requires the string field theory of the underlying $(2, 1)$ string.

In this paper we will start to examine the target space theory by understanding the constraints on the σ-model geometry of the $N = (2, 1)$ string due to the worldsheet supersymmetry and left-moving $U(1)$ current algebra. Our attitude is that while the $(2, 1)$ string may be unwieldy for describing the quantum mechanics of the theory, one should use whatever structure is available to

gain insight; in particular, an understanding of which fields the $(2,1)$ string can couple to should lead to some understanding of the configuration space of the 3-brane theory. We will take a very small step in this direction, by deriving to $\mathcal{O}(\alpha'^0)$ the constraints on the target space fields due to the $(2,1)$ worldsheet supersymmetry and the additional left-moving $U(1)$ symmetry. Such constraints have been discussed previously (Hull 1986b, Dine and Seiberg 1986, Braden 1987); in all of these discussions, the internal sector of the heterotic string was described by left-moving fermions (as these have the most natural coupling to the gauge fields) and the left-moving supersymmetry was imposed at tree level. These authors thus found that was that the internal fermions did not propagate and the gauge field was forced to be trivial. However, it is known that supersymmetry may be nontrivially realized on free fermions; in this case the algebra closes only at one loop (Goddard, Nahm and Olive 1985; Goddard, Kent and Olive 1986; Windey 1986; Antoniadis *et.al.* 1986; Gates, Howe and Hull 1989). In order to get the correct coefficients of the current commutators, the internal supersymmetry currents must scale as $1/\sqrt{\alpha'}$; therefore one-loop terms will appear at order $\mathcal{O}(\alpha'^0)$ in the commutator of the left-moving supersymmetry current G_- with itself. We will also find that $\mathcal{O}(\sqrt{\alpha'})$ corrections to the internal gauge fields (Equation (62)) and to the supersymmetry current (Equation (61)) mix with the internal supersymmetry currents to produce terms in $\{G_-, G_-\}$ at $\mathcal{O}(\alpha'^0)$. The result is that the internal theory can in fact be nontrivial in the presence of a left-moving superconformal symmetry.

The left-moving fermionic sector of the theory gives a nice geometrical structure (see also Braden (1987)). One may describe the left-moving fields of the $N = (2,1)$ string with 4 bosonic worldvolume coordinates and 28 fermions living in some vector bundle $\mathcal{V}$ fibered over the worldvolume. It is only after imposing supersymmetry that 4 of these fermions are associated with the left-moving excitations of the worldvolume coordinates and the gauge curvature in these directions is set equal to the worldvolume curvature. The identification of the tangent plane of the worldvolume can and will vary fiber by fiber in $\mathcal{V}$, as specified by the supersymmetry current; this identification will split $\mathcal{V}$ into "tangent" and "normal" bundles. Furthermore, the fibers of the "normal" bundle must live in the adjoint representation of some Lie group (which may be a product group); this group structure is encoded by the left-moving supersymmetry current as well. Thus the supersymmetry current adds some additional structure. Constraints on this structure and on the σ-model geometry are found in Equations (49), (50),(51),(60), (67), and (68). If we interpret the target space theory as describing the imbedding of a $2+2$-dimensional object in some spacetime, $\mathcal{V}$ should be related (nontrivially, as we will argue) to the spacetime.

More structure will arise as the $N = (2,1)$ string has a gauged, left-moving $U(1)$ supercurrent. We will find that the piece of the $U(1)$ current acting on the internal fermions must scale as $1/\sqrt{\alpha'}$, and so as with the supersymmetry algebra we will find that one-loop terms arise at order $\mathcal{O}(\alpha'^0)$, and that we will have to add a piece scaling as $\sqrt{\alpha'}$ to the supercurrent (Equation (107)) in order

to maintain consistency. The constraints on the geometry and the supercurrent to $\mathcal{O}(\alpha'^0)$ are shown in Equations (94),(75), (78), and (85). The upshot is that the target space possesses a null or timelike covariantly conserved Killing vector; if it is timelike, there must be an $\mathcal{O}(1/\sqrt{\alpha'})$ piece of the supercurrent acting on the internal fermions to cancel the anomaly of the target space isometry.

This thesis is organized as follows. Section 2 is a review of known pertinent results. Section 2.1 reviews the properties of $N = (2,1)$ strings with a flat target space; it also reviews the realization of supersymmetry on free fermions. Section 2.2 reviews the general σ-model action for the heterotic string and the constraints arising from $(2,0)$ supersymmetry as derived by Hull and Witten (1985). Section 3 describes the left-moving supersymmetry and $U(1)$ currents; it also lists the transformations of the fundamental σ-model fields under Dirac brackets with these currents. Section 4 contains the derivation of the constraints on the σ-model geometry arising from the left-moving supersymmetry. Section 5 contains the constraints due to the left-moving $U(1)$ supercurrent. Section 6 discusses of the physical and geometric interpretation of our results in terms of a 4-dimensional surface living in some spacetime. Section 7 contains conclusions and a discussion of directions for further work. Appendix A contains conventions for worldsheet and target space geometry. Appendix B describes the Dirac brackets of the $N = (2,1)$ σ-model. Finally, Section C contains conventions for the worldsheet Green's functions.

A note about nomenclature: in this paper the *right*-moving sector of the $N = (2,1)$ string will be the $N = 2$ sector and the *left*-moving sector will be the $N = 1$ sector. This is the convention used by Kutasov and Martinec (1996) and Kutasov, Martinec and O'Loughlin (1996).

2 Background

2.1 The $N = (2,1)$ string in flat spacetime

In order to get a feeling for the $N = (2,1)$ string we will first review the flat space theory. Many features of the σ-model in arbitrary backgrounds will be generalizations of this simple case; also, the physical massless states of the theory should tell us which backgrounds to turn on.

Strings with gauged $N = (2,2)$ worldsheet supersymmetry have been studied by Ademollo *et.al.* (1976a,b); Fradkin and Tseytlin (1981, 1985a); D'Adda and Lizzi (1987); Green (1987); and Mathur and Mukhi (1987, 1988). Ooguri and Vafa (1990, 1991a) explored the target space geometry more systematically by exploring the scattering amplitudes for physical states and the target space effective action one could deduce from these amplitudes. They also constructed heterotic string theories with gauged $(2,0)$ and $(2,1)$ supersymmetry (1990,1991b);[1] we will review these constructions here.

[1] Fradkin and Tseytlin (1985a) and Green (1987) also described the $(2,0)$ and $(2,1)$ het-

The right-moving sector of these theories includes in addition to the reparameterization and supersymmetry ghosts a pair of ghosts corresponding to the $U(1)$ piece of the $N=2$ algebra (the complex structure). The critical dimension of the right-movers is 4, and $N=2$ SUSY requires a complex structure (Alvarez-Gaumé and Freedman 1981; Hull and Witten 1985), so we must have either $(4+0)$ or $(2+2)$ signature. We will choose the latter. The left-moving bosons must then also have two timelike directions (in particular, if the target space is noncompact); however, the $N=1$ superconformal symmetry will only remove negative-norm states arising from one timelike direction. Unitarity requires an additional left-moving $U(1)$ gauge symmetry in order to remove the remaining negative-norm states. This gauging increases the critical dimension of the left-moving sector by 2, as there will be Faddev-Popov ghosts associated with the $U(1)$. Thus the matter sector of the left-movers of the $(2,0)$ string has $c=28$; the matter sector of the left-movers of the $N=(2,1)$ string has $\hat{c}=12$.[2]

Since the left-moving $U(1)$ current is gauged, nilpotency of the BRST charge requires that this current must have a vanishing operator product with itself (Ooguri and Vafa 1991b). In the case of the $N=(2,0)$ string in flat space, the $U(1)$ current will have the form

$$J_- = v_\mu \partial_- \phi_L^\mu \ , \tag{1}$$

where ϕ_L denote both the left-moving target worldvolume coordinates and the chiral scalars of the internal sector. It is easy to see that the condition for the vanishing OPE of this current with itself is $v_\mu v^\mu = 0$.

The internal sector of the $N=(2,0)$ string can be represented by 24 chiral scalars which live in the maximal torus of a rank 24 group. The physical states of the theory depend on the direction of the left-moving $U(1)$, i.e. on whether the vector has any components in the internal direction. With no components of the $U(1)$ in the internal direction the null isometry in $2+2$ dimensions kills off the gravitational dynamics of the target worldvolume gravitational dynamics, so that the only excitations come from the internal sector and correspond to the gauge bosons of the internal gauge group. Scattering amplitudes indicate that the target spacetime theory is the self-dual Yang-Mills theory in $2+2$ dimensions, reduced to a $1+1$-dimensional theory by the isometry. If the $U(1)$ has components in the internal direction, then the spacetime isometry is timelike and so the target spacetime is effectively $2+1$-dimensional. In this case there is some spacetime boson which is a remnant of the boson describing self-dual gravity in the $(2,2)$ string (Ooguri and Vafa 1990, 1991a). There are also the internal gauge fields as before; the gauge symmetry is partly broken by the internal part of the $U(1)$ (Ooguri and Vafa 1991b). In this case the theory describes some sort of coupling of self-dual Yang-Mills and self-dual gravity.

erotic string. In both papers the necessity of the left-moving $U(1)$ current was missed, although in the latter a truncation to a 2-dimensional target space was assumed.

[2] $\hat{c}=1$ for a free chiral superfield, or equivalently for three free Majorana- Weyl fermions.

For the $N = (2,1)$ string Ooguri and Vafa represented the internal sector by eight chiral scalars and their fermionic superpartners. They found that the only massless states correspond to the 8-dimensional ground state of the Ramond sector of the internal fermions and to the 8 Neveu-Schwarz states created by single fermion oscillators acting on the vacuum. Ooguri and Vafa therefore claimed that the internal sector has no group structure. However, one can find other realizations of the internal sector which do contain gauge symmetry. For example, Pierce (1996) used a free fermion construction of the internal SCFT, and found that the massless states in the adjoint of a 24-dimensional Lie algebra with the fermions living in the adjoint of this group. As we will use the fermionic representation of the $(2,1)$ σ-model below, let us review supersymmetric theories of free Majorana-Weyl fermions.

Given a theory with free left-moving Majorana-Weyl fermions λ^a, one may realize an affine Lie algebra with fermion bilinears (for a review and references see Goddard and Olive 1986). If we let the fermions live in the adjoint representation of a Lie group H, one may realize a worldsheet current algebra corresponding to this group:

$$J^a = f^{abc} \lambda^b \lambda^c \ , \tag{2}$$

where the coefficients f^{abc} are the structure constants of the H. There is a well-known $N = 1$ supersymmetry in such theories (Goddard, Nahm and Olive 1985; Goddard, Kent and Olive 1986; Windey 1986; Antoniadis *et.al.* 1986), where the supersymmetry charge is

$$G = \frac{1}{3\sqrt{C_A}} f^{abc} \lambda^a \lambda^b \lambda^c \ . \tag{3}$$

This realization with fermions living in the adjoint of a group is general. Following Windey (1986) and Antoniadis *et.al.* (1986) we can realize $N = 1$ supersymmetry on a set of free Majorana-Weyl fermions by splitting them up into fermions living in the adjoint of some group H and fermions living in some representation of that group. Requiring that the superconformal algebra closes appropriately means that sum of these spaces is in fact the adjoint representation of some group $G \supset H$ and that G/H is a homogeneous space. The level of the current algebra is the dual Coxeter number of this group. Ooguri and Vafa (1991b) used the vertex operator construction of the current algebra (Halpern 1975; Frenkel and Kac 1980; Segal 1981; Gross *et.al.* 1985, 1986) which is a level 1 construction. If we fermionized this theory we would find that the only current algebra compatible with left-moving supersymmetry at this level is $U(1)^8$.

If we wish to find which states correspond to gauge bosons and what the gauge structure is, we need to choose an appropriate GSO projection compatible with modular invariance. The spectrum of massless states and its group structure depends on the choice of SUSY current and on the GSO projection (Pierce 1996), just as in the free fermion models of type II string theories with gauge

symmetry (Kawai, Llewellyn and Tye 1986a,b and 1987a,b; Bluhm, Dolan and Goddard 1987). In particular the construction of different string and membrane theories as target spaces of the $(2,1)$ string relies on the spectra one gets from different GSO projections (Kutasov and Martinec 1996).

It is also worth recalling the conditions for target space supersymmetry. Since the right-moving part of the theory has gauged $N=2$ supersymmetry, the spectral flow is gauged; thus the Neveu-Schwarz and Ramond sectors of the right-movers are equivalent (Schwimmer and Seiberg 1987; Ooguri and Vafa 1991b). Spacetime supersymmetry must arise entirely from the left-movers; however, a well-known theorem of Friedan and Shenker (unpublished; for a published description and discussion see Section 2.4 of Dixon, Kaplunovsky and Vafa (1987)) states that target space supersymmetry and non-Abelian gauge symmetry cannot arise from the same sector. Equivalently, target space supersymmetry exists if and only if the left-moving sector possesses global $N=2$ supersymmetry (Banks *et.al.* 1988; Banks and Dixon 1988). Thus, such a global supersymmetry is also incompatible with non-Abelian gauge symmetry. Indeed, the constructions of Kutasov and Martinec (1996) and Kutasov, Martinec and O'Loughlin (1996) which possess spacetime supersymmetry have no non-Abelian structure.

2.2 The σ-model action, and constraints from $(2,0)$ supersymmetry

We are interested in the more general σ-model which one would find upon condensing vertex operators corresponding to the massless bosonic physical states of the $(2,1)$ string. We know that in general the bosonic sector contains target space gravity and gauge fields. We will also include an antisymmetric tensor field, as torsion arises naturally in heterotic σ-models: since no antisymmetric tensor field appears in the physical state spectrum this field will merely dress the gauge and gravitational excitations in solutions to the β-function equations.

We can break up the fields of the $N=(2,1)$ sigma model into several pieces: d target worldvolume coordinates which we can write as $N=(2,0)$ superfields, making the right-moving supersymmetry manifest (Dine and Seiberg, 1986); d left-moving Majorana-Weyl fermions to pair with the left-moving target space bosons; and a piece describing the internal left-moving conformal field theory. The internal theory can be written as a set of $3n$ real, left-moving fermions with indices in some $3n$-dimensional tangent space, coupled to a background vector potential, as a set of n left-moving chiral scalars paired to n left-moving real fermions by the $N=(0,1)$ SUSY, or as any other left-moving $\hat{c}=8$ superconformal field theory; we will work with the fermionic representation, since the coupling to target space gauge fields is straightforward.

The action for the target space bosons and their right-moving fermionic superpartners may be written in $(2,0)$ superspace following Dine and Seiberg

(1986):

$$S_{st} = -\frac{i}{2}\int d^2\sigma d\theta d\theta^* \left[K_i(\Phi^j, \Phi^{\bar{k}})\partial_-\Phi^i - K_{\bar{i}}(\Phi^j, \Phi^{\bar{k}})\partial_-\Phi^{\bar{i}}\right] \ . \tag{4}$$

where we use the notation given in Appendix A. The action in component form is (Hull and Witten 1985):

$$\begin{aligned} S_{st} = \frac{1}{2}\int d^2\sigma \, [&(g_{\mu\nu}(\phi) + b_{\mu\nu}(\phi))\, \partial_+\phi^\mu\partial_-\phi^\nu \\ &+ig_{\mu\nu}\psi^\mu \left(\partial_-\psi^\nu + \Gamma_{(+)}{}^\nu_{\lambda\rho}(\phi)\partial_-\phi^\lambda\psi^\rho\right)\Big] \ . \end{aligned} \tag{5}$$

We will work with this form, dropping the manifestly complex parameterization; there will be a complex structure tensor J^μ_ν with the properties required for $(2,0)$ supersymmetry, which we will discuss below.

We could write the left-moving fermions in $(2,0)$ superspace as in Dine and Seiberg. This would force a Hermitean structure on the vector bundle which the internal fermions live in, as required for off-shell closure of the $N = 2$ algebra (Howe and Papadopolous 1988). We will worry only about on-shell closure in this paper. In $(1,0)$ superspace the action for left-moving Majorana fermions coupled to a background gauge field is (Hull and Witten 1985):

$$S_{\mathrm{Rf}} = -\int d^2\sigma d\theta_+ G_{ab}(\Phi)\Lambda^a \left(D_+\Lambda^b + A_\mu{}^b{}_c D_+\Phi^\mu\Lambda^c\right) \ . \tag{6}$$

In this equation Φ is the $(1,0)$ bosonic superfield as written in Appendix A. Integrating over the Grassman coordinate θ_+ and eliminating the auxiliary fields, we find the component form of the action (Hull and Witten 1985):

$$S_{Lf} = \frac{1}{2}\int d^2\sigma \left[iG_{ab}(\phi)\lambda^a \left(\partial_+\lambda^b + A_\mu{}^b{}_c\partial_+\phi^\mu\lambda^c\right) + \frac{1}{2}F_{\mu\nu ab}\psi^\mu\psi^\nu\lambda^a\lambda^b\right] \ . \tag{7}$$

One may rotate away the metric G with the vielbein fields $\rho^a{}_A$ and obtain the action

$$\begin{aligned} S_{Lf} = \frac{1}{2}\int d^2\sigma \, \Big[&i\eta_{AB}(\phi)\lambda^A \left(\partial_+\lambda^B + \omega_\mu{}^B{}_C\partial_+\phi^\mu\lambda^C\right) \\ &+\frac{1}{2}F_{\mu\nu AB}\psi^\mu\psi^\nu\lambda^A\lambda^B\Big] \ . \end{aligned} \tag{8}$$

There will also be a dilaton term appearing appearing in the action at $\mathcal{O}(\alpha')$; in conformal gauge this should couple to the ghosts as described by Banks, Nemeschansky and Sen (1986). One may redefine the ghost fields in such a way that the dilaton does not appear in the action; this transformation is anomalous, so the BRST current picks up a term proportional to the dilaton at order α'

(for the superstring version of this in superspace, see Aldazabal, Hussain and Zhang (1987)). This is of too high an order to appear in our $\mathcal{O}(\alpha'^0)$ calculation.

The fermions live in a vector bundle $\mathcal{V}$ with connection $\hat{A}_\mu{}^a{}_b$ fibered over the target space (Hull and Witten 1985). We expect the left-moving supersymmetry to relate some 4-plane in each fiber to the tangent space of the worldvolume; it should also somehow relate the "tangent" gauge connection to the spin connection (indeed, we will find that the curvatures agree). We will not make this identification at the level of the action; rather, all of the information encoding this identification will lie in the supersymmetry current. The advantages of this presentation will become clear when we discuss the interpretation of target space theory as a 4-dimensional worldvolume immersed in some spacetime.

The combination of actions (5) and (7) is invariant under the supersymmetry transformation [3]

$$\begin{aligned} \delta\phi^\mu &= \epsilon\psi^\mu && (9)\\ \delta\psi^\mu &= i\epsilon\partial_+\phi^\mu && (10)\\ \delta\lambda^a &= -\epsilon\hat{A}_\mu{}^a{}_c\psi^\mu\lambda^c && (11) \end{aligned}$$

and the complex structure transformation

$$\delta\psi^\mu = J^\mu{}_\nu\psi^\nu\ ,\ \ \delta\phi = \delta\lambda = 0\ . \tag{12}$$

The second supersymmetry comes by commuting the first supersymmetry with the complex structure (Hull and Witten 1985). Invariance of equations (5) and (7) under the complex structure rotation (12) combined with closure of the $N = 2$ algebra place requirements on the complex structure (Hull and Witten 1985):

$$\begin{aligned} &J^\mu_\beta J^\beta_\nu = -\delta^\mu_\nu && (13)\\ &N^\mu_{\nu\rho} = J^\beta_\nu\partial_{[\beta}J^\mu_{\rho]} - J^\beta_\rho\partial_{[\beta}J^\mu_{\nu]} = 0 && (14)\\ &g_{\alpha\beta}J^\alpha_\mu J^\beta_\nu = g_{\mu\nu} && (15)\\ &\nabla_{(+)\lambda}J^\mu_\nu = 0 && (16)\\ &J^\mu{}_{[\nu}F_{\lambda]\mu ab} = 0\ . && (17) \end{aligned}$$

Note also that the combination of Equations (14) and (16) leads to an algebraic constraint on the torsion (Delduc, Kalitzin and Sokatchev 1990):

$$H^\mu{}_{\nu\rho} - H^\mu{}_{\lambda\tau}J^\lambda{}_\tau J^\tau{}_\rho + H^\tau{}_{\nu\lambda}J^\mu{}_\tau J^\lambda{}_\rho + H^\tau{}_{\lambda\rho}J^\mu{}_\tau J^\lambda{}_\nu = 0\ . \tag{18}$$

Any two of Equations (14), (16), (18) are independent. One may also rewrite Equations (13) - (18) as the vanishing of various components of the connection,

[3] In the paper of Hull and Witten (1985) the sign in the variation of λ in Equation (11) is incorrect. Braden (1987) also uses the sign opposite that of Equation (11), but in his paper the sign is correct as the action for ψ is defined with a total sign opposite that of the action (7).

torsion, and curvature in complex coordinates (Hull and Witten 1985, Bonneau and Valent 1994).

3 Construction of SUSY and $U(1)_L$ currents

We will describe the left-moving $N = 1$ supersymmetry and $U(1)$ via their current algebra. Although we will work only to $\mathcal{O}(\alpha'^0)$, the supersymmetry is realized quantum mechanically on the internal fermions and the algebra only closes to this order after including worldsheet loop effects in the commutators.

We begin by building the supersymmetry current out of the most general terms which have dimension $(0, 3/2)$ and the $U(1)$ current out of terms which have dimension $(0, 1)$. We then ask that the algebra close properly and that the variation of the action under the classical transformations vanish up to the divergence of the currents, or, integrating by parts, up to terms of the form

$$\partial_- \epsilon J_+ + \partial_+ \epsilon J_- \ , \tag{19}$$

where ϵ parameterizes the variation, $\delta\xi = \{\xi, \epsilon J\}$. Since we want left-moving currents we also demand that J_+ vanish.

To order α'^0 the equal-time commutators are usually just the tree-level Poisson or Dirac brackets of the currents. However, as we have discussed above, supersymmetry is realized nonlinearly and quantum mechanically on the internal fermions. It is easy to see that the operator product of the supersymmetry current (3) with itself is:

$$G(w_-)G(z_-) \sim i\,(\alpha')^2\,\frac{d}{12}\frac{1}{(w_- - z_-)^3} - \alpha'\frac{1}{(w_- - z_-)}\eta_{ab}\lambda^a\partial_-\lambda^b \ , \tag{20}$$

where we have explicitly included the α' factors coming from fermion loops. This is the correct operator product for the left-moving SUSY charge of a system with central charge $\hat{c} = d/3$ within a multiplicative factor of α' (for a discussion of this factor in path integral language see (Gates, Howe and Hull 1989)). To get operator product coefficients of the superconformal algebra we must rescale G:

$$G = \frac{1}{3\sqrt{\alpha' C_A}} f_{abc}\lambda^a\lambda^b\lambda^c \ . \tag{21}$$

One may also see this scaling by starting with the left-moving supersymmetry current of a scalar field plus its fermionic superpartner:

$$G_L = \partial_-\phi\lambda^1 \ . \tag{22}$$

Now fermionize the scalar:

$$\sqrt{\frac{\alpha'}{2}}\partial_-\phi =: \lambda^2\lambda^3 : \ . \tag{23}$$

The coefficient on the left-hand side of this equation, including the power of α', can be fixed by matching the two-point function of each side. Substituting this into (22) will give us a supersymmetry current like (21). Because of this α' dependence, we must be a bit careful in counting orders of α' in our computation; for example, one loop terms in operator products of the above supersymmetry charge will enter at order ${\alpha'}^0$.

In previous discussions of σ-models with $(2,1)$ supersymmetry (Hull 1986b; Dine and Seiberg 1986; Braden 1987) closure of the algebra was imposed at tree level. This forces the gauge fields to be flat and the internal fermions to be non-propagating. However, in flat space there are physical vertex operators corresponding to gauge field fluctuations; nothing prevents us from condensing these operators on the worldsheet as long as their expectation values satisfy the β-function equations. The latter three authors listed above argued that because supersymmetry pairs bosons and fermions, the internal fermions must be trivial since they are not paired with bosonic fields. However, in the full quantum treatment we may have a representation of supersymmetry on the Fock space of the system which does not close on the one-particle states.

With these arguments in mind, we also expect that the part of the left-moving $U(1)$ composed of left-moving fermions will have a piece quadratic in fermions and scaling as $1/\sqrt{\alpha'}$, $\Sigma_{ab}\lambda^a\lambda^b$. This is especially necessary for the $U(1)$ which leads to membrane constructions (Kutasov and Martinec 1996; Kutasov, Martinec and O'Loughlin 1996). For these constructions the part of the $U(1)$ living in the target space is timelike; for example, if the current in flat space looks like:

$$J_- = v_\mu \partial_- \phi^\mu + \Sigma_{ab}\lambda^a\lambda^b \ , \tag{24}$$

and $v^2 < 0$, then the anomaly in the operator product of this with current can only vanish by setting $\alpha'\Sigma_{ab}\Sigma^{ab} + v_\mu v^\mu = 0$, where the first term is a one-loop term. Thus, Σ must generally have a piece scaling as $1/\sqrt{\alpha'}$.

3.1 Review of $(2,0)$ SUSY charges

The right-moving currents for $N = 2$ SUSY can be written down most easily as $(1,0)$ superfields (see for example Hull and Spence 1990):

$$\begin{aligned} \mathcal{G}^{(1)}_+ &= g_{\mu\nu}(\Phi)\, D_+\Phi^\mu \partial_+\Phi^\nu - \frac{i}{6} H_{\mu\nu\rho}(\Phi)\, D_+\Phi^\mu D_+\Phi^\nu D_+\Phi^\rho & (25)\\ \mathcal{U}_+ &= \frac{i}{2} J_{\mu\nu}(\Phi)\, D_+\Phi^\mu D_+\Phi^\nu \ . & (26)\end{aligned}$$

Expanding the superfields in components we find (see Appendix A for notation):

$$\begin{aligned} \mathcal{G}^{(1)}_+ &= g_{\mu\nu}\psi^\mu\partial_+\phi^\nu - \frac{i}{6}H_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho \\ &\quad + i\theta_+\left\{ g_{\mu\nu}\partial_+\phi^\mu\partial_+\phi^\nu + i g_{\mu\nu}\psi^\mu\left(\partial_+\psi^\nu + {\Gamma_{(+)}}^\nu_{\lambda\rho}\partial_+\phi^\lambda\psi^\rho\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= G_+^{(1)} + i\theta T_{++} &(27)\\
\mathcal{U}_R &= \frac{i}{2} J_{\mu\nu}\psi^\mu\psi^\nu + \theta\left\{ J_{\mu\nu}\psi^\mu\partial_+\phi^\nu + \frac{i}{2}\partial_\rho J_{\mu\nu}\psi^\mu\psi^\nu\psi^\rho \right\}\\
&= J_+ + \theta G_+^{(2)} \ . &(28)
\end{aligned}
$$

Here $G_+^{(1)}$, $G_+^{(2)}$ are the two right-moving supersymmetry currents, T_{++} is the right-moving stress-tensor, and J_+ is the $U(1)$ part of the right-moving superconformal algebra. It is easy to see, using the fundamental Dirac brackets described in Appendix B, that the above currents generate the transformations listed in Equations (9)-(12).

3.2 The left-moving currents

The most general dimension $(0, 3/2)$ operator we can write down is:

$$
G_- = e_{\mu a}\partial_-\phi^\mu\lambda^a - i f_{abc}\lambda^a\lambda^b\lambda^c \ . \tag{29}
$$

The variations of the component fields induced by Dirac brackets with this current are:

$$
\begin{aligned}
\delta\phi^\mu &= \epsilon\, e^\mu{}_a\lambda^a &(30)\\
\delta\psi^\mu &= \epsilon\, e^\rho{}_a\Gamma_{(+)}{}^\mu_{\rho\alpha}\psi^\alpha\lambda^a &(31)\\
\delta\lambda^a &= i\epsilon e_\mu{}^a\partial_-\phi^\mu + \epsilon\left(3f^a{}_{bc} - e^\rho{}_b\hat{A}_\rho{}^a{}_c\right)\lambda^a\lambda^b\\
&\equiv i\epsilon e_\mu{}^a\partial_-\phi^\mu + \epsilon B^a{}_{bc}\lambda^b\lambda^c &(32)
\end{aligned}
$$

The object $e_{\mu a}$ maps the 4-dimensional tangent bundle of the target worldvolume into the 28-dimensional vector bundle $\mathcal{V}$. We may also form the projector (Braden 1987)

$$
\mathcal{P}^a{}_b = \delta^a{}_b - e_\mu{}^a e^\mu{}_b \ . \tag{33}
$$

This projects indices onto what we will call the normal part of $\mathcal{V}$ while $(1-\mathcal{P})$ projects indices onto the tangent part of $\mathcal{V}$: if e has maximal rank (which should be true for generic points in the target worldvolume), the normal part of the bundle will have 24-dimensional fibers and the tangent part will have 4-dimensional fibers. In flat space, as we have discussed, the fermions will lie in the adjoint representation of some Lie algebra and the normal part of f^{abc} will be the structure constants of the algebra. We expect that this structure will persist fiberwise in the presence of nontrivial $g_{\mu\nu}$, $\hat{A}$, and e.

The most general dimension $(0, 1)$ operator is:

$$
J_- = v_\mu\partial_-\phi^\mu + \Sigma_{ab}\lambda^a\lambda^b \ , \tag{34}
$$

where all the coefficients depend on ϕ as usual. The variations of the σ-model variables under this current are:

$$\delta\phi^\mu = \epsilon\, v^\mu \tag{35}$$

$$\delta\psi^\mu = -\epsilon\, \Gamma_{(-)}{}^\mu_{\nu\rho} v^\rho \psi^\nu \tag{36}$$

$$\delta\lambda^a = -\epsilon\, \hat{A}_\rho{}^a{}_b v^\rho \lambda^b - 2i\Sigma^a{}_b \lambda^b \ . \tag{37}$$

v and Σ will be restricted by demanding chirality, invariance of the action under the above variations, and closure of the left-moving algebra.

4 Imposing $N = (0,1)$ SUSY

As with $(2,2)$ (Alvarez-Gaumé and Freedman 1981) and $(2,0)$ (Hull and Witten 1986) supersymmetry in 2d σ-models, $(2,1)$ supersymmetry places constraints on the geometry of the target space fields. We find our constraints in the usual fashion, by demanding the classical invariance of the action under the variations (30)-(32), and closure of the supersymmetry algebra. The constraints arising from the right-moving supersymmetries have been reviewed above. The constraints on fields coupled to the tangent fermions, were found by Braden (1987); we will rederive his results in the course of our analysis.

In the variation of the action we can clearly separate the terms into pieces of different order in the left- and right-moving fermions and their derivatives. The bosonic part of the action will give us all the terms linear in λ. After several integrations by parts and the use of the equations of motion for ϕ, we find that the term proportional to λ is:

$$\delta S = \int d^2\sigma \, \{ \partial_+ \epsilon\, e_{\mu a} \lambda^a \partial_- \phi^\mu + \epsilon \left[\partial_\mu e_{\nu a} - \hat{A}_\mu{}^b{}_a e_{b\nu} - \Gamma_{(+)}{}^\alpha_{\nu\mu} e_{\alpha a} \right] \lambda^a \partial_+ \phi^\mu \partial_- \phi^\nu \Big\} \ . \tag{38}$$

The first term gives us the first piece of the supersymmetry current (34) as required by Noether's theorem. The vanishing of the second term was interpreted by Braden (1987) as an equality between the spin and (tangent bundle) gauge connections up to a (sort of) gauge transformation. Another interpretation is that e is covariantly constant to $\mathcal{O}(\alpha'^0)$:

$$\hat{D}_{(-)\mu} e^\rho{}_a = 0 \ , \tag{39}$$

which implies that

$$\hat{D}_\lambda \mathcal{P}^a{}_b = 0 \tag{40}$$

at $\mathcal{O}(\alpha'^0)$. The terms proportional to $\psi\psi\lambda$ are:

$$\int i\epsilon \left[R_{(+)\mu\nu\lambda\rho} e^\lambda{}_a - F_{\mu\nu ab} e^b{}_\rho \right] \partial_- \phi^\rho \lambda^a \psi^\mu \psi^\nu \ . \tag{41}$$

This sets the tangent part of F_{ab} equal to $R_{(+)}$, and also implies that F_{ab} splits entirely into tangent and normal pieces. The term cubic in λ is:

$$T_{3\lambda} = \int d^2\sigma \left\{ -i\partial_+\epsilon f_{abc}\lambda^a\lambda^b\lambda^c \right. \\ \left. +i\epsilon\left(\frac{1}{2}e^\alpha{}_a F_{\alpha\rho bc} - \hat{D}_\rho f_{abc}\right)\partial_+\phi^\rho\lambda^a\lambda^b\lambda^c \right\} \ . \tag{42}$$

The first term multiplying $\partial_+\epsilon$ combines with the first term of the integrand in Equation (38) to form the supersymmetry current, as required by Noether's theorem. The second term will be discussed below. Finally, there is a term cubic in λ and quadratic in ψ, arising from the variation of the $F\psi\psi\lambda\lambda$ term in the action:

$$T_{2\psi 3\lambda} = \int d^2\sigma \left\{ \left(e^\rho{}_a \hat{D}^{(+)}_\rho F_{\mu\nu bc} + 6F_{\mu\nu ka}f^k{}_{bc}\right)\psi^\mu\psi^\mu\lambda^a\lambda^b\lambda^c \right\} \ . \tag{43}$$

Next, we wish to examine the constraints arising from closure of the supersymmetry algebra:

$$\{G_-(\sigma), G_-(\sigma')\} = T_{--}(\sigma') \ \delta(\sigma-\sigma') \tag{44}$$
$$\{G_-(\sigma), G_+^{(1)}(\sigma')\} = 0 \tag{45}$$
$$\{G_-(\sigma), J_+(\sigma')\} = 0 \tag{46}$$
$$\{G_-(\sigma), G_+^{(2)}(\sigma')\} = 0 \ . \tag{47}$$

Note that the last equation follows from the first three by the Jacobi identity. We will be interested in the $\mathcal{O}(\alpha'^0)$ part of these commutators. As discussed above the first of these commutators will have two pieces at order $\mathcal{O}(\alpha'^0)$: one will arise from the classical Dirac bracket and the other will arise from the one-loop contribution of the commutator of the cubic part of the supersymmetry current with itself. A long and tedious calculation, using the results above, reveals the classical Dirac brackets of the left-moving SUSY currents to be:

$$\{G_-(\sigma), G_-(\sigma')\} = \delta(\sigma-\sigma')\times \\ \Big\{ -ie_\mu{}^a e_{\nu a}\partial_-\phi^\mu\partial_-\phi^\nu + e_{\rho a}e^\rho{}_b\lambda^a\left(\partial_-\lambda^b + \hat{A}_\mu{}^b{}_c\partial_-\phi^\mu\lambda^c\right) \\ -\Big[\left(6f_{abc}e_\rho{}^c + H_{\alpha\beta\rho}e^\alpha{}_a e^\beta{}_b\right) \\ +2e^\mu{}_a\hat{D}_{(-)\mu}e_{\rho b} - e^\mu{}_a\hat{D}_{(-)\rho}e_{\mu b}\Big]\partial_-\phi^\mu\lambda^a\lambda^b \\ -e^\mu{}_a\hat{D}_{(-)\rho}e_{\mu b}\partial_+\phi^\rho\lambda^a\lambda^b \\ +\frac{i}{2}\left(R_{(+)\lambda\rho\alpha\beta}e^\alpha{}_a e^\beta{}_b - e_{\mu a}e^\mu{}_k F_{\lambda\rho}{}^k{}_b\right)\lambda^a\lambda^b\psi^\lambda\psi^\rho \\ +i\left(\frac{1}{2}e^\alpha{}_a e^\beta{}_b F_{\alpha\beta cd} + 2e^\alpha{}_a\hat{D}_\alpha f_{bcd} + 9f^k{}_{ab}f_{kbc}\right)\lambda^a\lambda^b\lambda^c\lambda^d \Big\} \ . \tag{48}$$

Note that we had to use the equations of motion for λ in order to get terms with τ derivatives of λ. The first line of (48) is clearly the stress tensor of the bosons and of the fermions tangent to the target worldvolume, if

$$e_\mu{}^a e_{\nu a} = g_{\mu\nu} \ . \tag{49}$$

We have kept the $\hat{D}_{(-)}e$ terms in the fourth and fifth line. They vanish to this order, but we will want to discuss order $\mathcal{O}(\sqrt{\alpha'})$ corrections below. To order $\mathcal{O}(\alpha'^0)$ the vanishing of the third line of (48) requires f to split into a completely normal piece $f^\perp$ and a completely tangent piece:

$$f^\parallel{}_{abc} = -\frac{1}{6} e^\alpha{}_a e^\beta{}_b e^\gamma{}_c H_{\alpha\beta\gamma} \ . \tag{50}$$

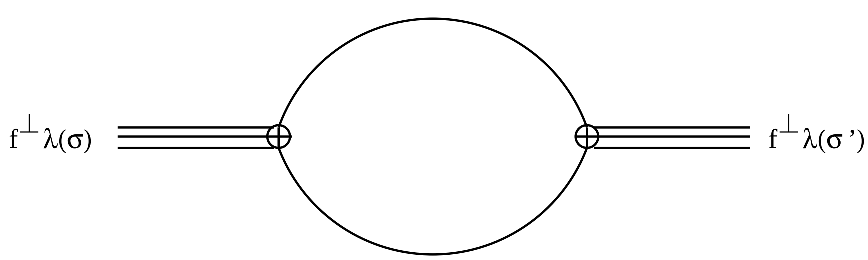
As we will see shortly, corrections of this splitting at $\mathcal{O}(\sqrt{\alpha'})$ mix into the last line of Equation (48). The vanishing of the sixth line follows from the vanishing of (41). The final line has a piece of order $1/\alpha'$. If we were to think of f as some structure constants on the normal part of the fibers of the bundle V, then the vanishing of this term,

$$f^{(\perp)}{}_{k[ab} f^{(\perp)k}{}_{cd]} = 0 \ , \tag{51}$$

enforces the Jacobi identity on the structure constants. Note that if f receives an $\mathcal{O}(\sqrt{\alpha'})$ correction $f^{(1)}$, then the ff term will have an additional $\mathcal{O}(\alpha'^0)$ term coming from $f^\perp f^{(1)}$. At order $\mathcal{O}(1/\sqrt{\alpha'})$,

$$\hat{D}_\lambda f^\perp{}_{abc} = 0 \ . \tag{52}$$

This equation will have $\mathcal{O}(\alpha'^0)$ corrections.



+ terms covariantizing derivatives

Figure 1

The one-loop $\mathcal{O}(\alpha'^0)$ contribution to the commutator $\{G_-, G_-\}$. The single straight lines denote λ propagators; the triple lines denote background field insertions. Crossed circles denote vertices arising from operator insertions.

We must also examine the one loop contribution to the commutator of $f^{(\perp)}{}_{abc}\lambda^a\lambda^b\lambda^c$ with itself. This can be calculated by expanding the operators and the action in Riemannian normal coordinates (Alvarez-Gaumé, Freedman and Mukhi 1981; Sen 1985; Banks, Nemeschansky and Sen 1986). We can get away with calculating a subset of the resulting terms since the normal coordinate expansion is covariant with respect to both the target space gauge and coordinate indices. Terms coming from expansions of f in normal coordinates will involve derivatives of f which will be covariantized; as argued above, such terms will vanish at $\mathcal{O}(\alpha'^0)$ (although not necessarily at $\mathcal{O}(\sqrt{\alpha'})$). Terms in the commutator involving the gauge and spin connection will either covariantize derivatives or form combinations and covariant derivatives of the appropriate curvature tensors.

In general this argument is too naive. The σ-model anomaly (Moore and Nelson 1984,1985; Hull and Witten 1985) spoils invariance with respect to local Lorentz and gauge transformations of the background fields. This lack of gauge invariance can be absorbed by an anomalous variation of $b_{\mu\nu}$ and a redefinition of the antisymmetric tensor field strength (Callan *et.al.* 1985; Hull and Witten 1985; Sen 1986; Hull and Townsend 1986). At any rate, the σ-model anomaly will not show up to the order we are concerned with. Another potential problem arises if we expand the action around a background field configuration which does not satisfy the equations of motion; in this case, the normal coordinate expansion of the action will include noncovariant terms proportional to the classical equations of motion (Alvarez-Gaumé, Freedman and Mukhi 1981). This leads to noncovariant divergences in the action which are removed by wavefunction renormalization. Such terms should be included when renormalizing the theory, but once this is done we can calculate the renormalized Green's functions by expanding the action around solutions to the classical equations of motion.

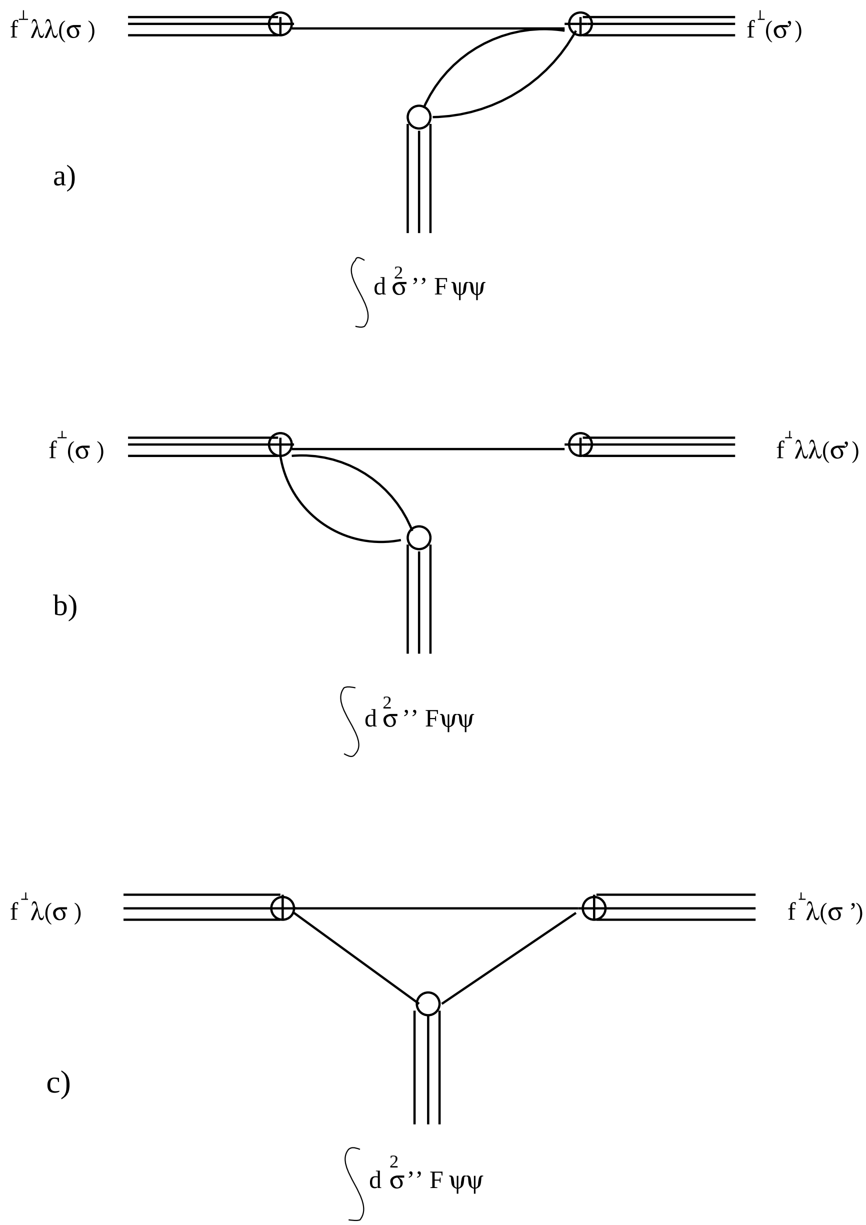


Figure 2

The remaining one-loop $\mathcal{O}(\alpha'^0)$ diagrams, after Figure 1. The open circles denote vertices arising from the interaction part of the Lagrangian.

The relevant one-loop Feynman diagrams contributing to the commutator are shown in Figures 1 and 2. The diagram in Figure 1 will give us a coefficient

of $1/(x_- - y_-)^2$ times a bilocal operator: expanding the operator around y gives us a term multiplying $1/(x_- - y_-)$ which will contain derivatives of f and λ which will become covariantized. The diagrams in Figure 2 cancel each other due to the Jacobi identity (51). The result for the one-loop $\mathcal{O}(\alpha'^0)$ operator product is:

$$G_-(x_-)G_-(y_-) = -\frac{18}{4}\frac{\alpha'^2}{(x_- - y_-)^2} f^{(\perp)}{}_A{}^{CD} f^{(\perp)}{}_{BCD}\lambda^A\lambda^B(y_-) + \frac{18}{4}\frac{\alpha'}{(x_- - y_-)} f^{(\perp)}{}_A{}^{CD} f^{(\perp)}{}_{BCD}\lambda^A\left(\partial_-\lambda^B + \omega_\mu{}^B{}_K\partial_-\phi^\mu\lambda^K\right) \ . \tag{53}$$

The last term in the operator product is equal to $iT^{(\perp)}/2(x_- - y_-)$, as required for closure, provided that

$$f^{(\perp)}{}_{ACD} f^{(\perp)}{}_B{}^{CD} = -\frac{1}{9\alpha'}\mathcal{P}^K{}_A\mathcal{P}^L{}_B\eta_{KL} \ , \tag{54}$$

This equation and Equation (51) indicates that the coefficients $f^{(\perp)}$ are proportional to structure constants of a Lie algebra. Equation (54) also insures that the $1/(x_- - y_-)^2$ term vanishes, and gives the $1/(x_- - y_-)^3$ term expected for 24 free fermions. The remaining singular part of the operator product can be converted to an expression for the equal-time commutator in the usual fashion (see Appendix C).

If Equation (54) satisfied we may add to Equation (48) the term

$$\mathcal{P}^k{}_a\mathcal{P}_{kb}\lambda^a\left(\partial_-\lambda^b + \hat{A}_\mu{}^b{}_c\lambda^c\right) \tag{55}$$

at order $\mathcal{O}(\alpha')$. Adding this term to the second line of Equation (48) should give us T_{--} as required for closure.

Another long calculation reveals that to $\mathcal{O}(\alpha'^0)$

$$\{G_-(\sigma), G_+(\sigma')\} = i\left(\frac{1}{2}e^\alpha{}_a F_{\alpha\rho bc} - \hat{D}_\rho f_{abc}\right)\psi^\rho\lambda^a\lambda^b\lambda^c \ . \tag{56}$$

The vanishing of this term follows from the vanishing of (42). Note that since there are no terms with negative powers of $\sqrt{\alpha'}$ appearing in G_+ all the terms in (56) will arise at tree level.

Now we can discuss the solution to the constraints we have derived. The operator product contains no piece of $G_{ab}\lambda^a\partial_-\lambda^b$ where G has both tangent and normal indices: in other words, the metric is block diagonal with respect to the splitting of $\mathcal{V}$ defined by $\mathcal{P}$. In addition, the vanishing of (41) indicates that F is also block diagonal in its gauge indices at $\mathcal{O}(\alpha'^0)$. If we split F into $F^{(\|)}$ and $F^{(\perp)}$, and if we let $f = f^{(\perp)} - iH/6$, we find that using the Bianchi identities in the appropriate manner, terms involving $F^{\|}$ and H drop out of the the last

line of Equations (48), (42), and (43). This leaves the following constraints:

$$\left\{\frac{1}{2}F^{(\perp)}{}_{\mu\rho bc}e^{\mu}{}_{a} - \hat{D}_{\rho}f^{(\perp)}{}_{abc} = 0\right\}\lambda^a\lambda^b\lambda^c \tag{57}$$

$$\left\{\frac{1}{2}e^{\alpha}{}_{a}e^{\beta}{}_{b}F^{(\perp)}{}_{\alpha\beta cd} + 2e^{\alpha}{}_{a}\hat{D}_{\alpha}f^{(\perp)}{}_{bcd} + 9f^{(\perp)}_{kab}f^{(\perp)k}{}_{cd}\right\}\lambda^a\lambda^b\lambda^c\lambda^d = 0 \tag{58}$$

$$\left\{\hat{D}_{(+)\rho}F^{(\perp)}_{\mu\nu bc}e^{\rho}{}_{a} + 6F^{(\perp)}_{\mu\nu ka}f^{(\perp)k}{}_{bc}\right\}\lambda^a\lambda^b\lambda^c = 0 \ . \tag{59}$$

It seems at first glance impossible to solve these constraints for nontrivial $F^{(\perp)}$. However, order $\mathcal{O}(\sqrt{\alpha'})$ terms in the supersymmetry current combine with order $\mathcal{O}(1/\sqrt{\alpha'})$ terms in the above equations and cancel off the nontrivial field strengths. Closure to order $\mathcal{O}(\sqrt{\alpha'})$ requires that one-loop terms in the commutator cancel the classical $\mathcal{O}(\sqrt{\alpha'})$ terms: we will calculate one of these terms below.

Let us assume that f has a term $f^{(1)}$ scaling as $\sqrt{\alpha'}$ with one normal index and two tangent indices. If we substitute Equation (58) into Equation (57), and use the version of Equation (54) with curved indices,

$$f^{(\perp)}{}_{akl}f^{(\perp)}{}_{b}{}^{kl} = -\frac{1}{9}G_{ab} \ , \tag{60}$$

we find that

$$f^{(1)}_{abk}\mathcal{P}^{k}{}_{c} = -\frac{1}{4}\alpha' e^{\alpha}{}_{a}e^{\beta}{}_{b}F_{\alpha\beta kl}f^{(\perp)}{}_{c}{}^{kl} \ . \tag{61}$$

We may solve Equation (59) by adding an order $\mathcal{O}(\sqrt{\alpha'})$ piece $\hat{A}^{(1)}$ to the gauge connection $\hat{A}$:

$$\hat{A}^{(1)}{}_{\mu}{}^{a}{}_{b}e^{b}{}_{\rho} = \frac{3}{2}\alpha' F^{(0)}{}_{\rho\mu kl}f^{(\perp)akl} \ , \tag{62}$$

where $F^{(0)}$ is the curvature for the uncorrected connection $\hat{A} - \hat{A}^{(1)}$. If $\hat{D}^{(0)}$ is the gauge-covariant derivative with respect to the uncorrected connection, then the order $\mathcal{O}(\sqrt{\alpha'})$ correction to the curvature is:

$$F_{\mu\nu}{}^{a}{}_{b} = F^{(0)}{}_{\mu\nu}{}^{a}{}_{b} + \hat{D}^{(0)}{}_{\mu}\hat{A}^{(1)}{}_{\nu}{}^{a}{}_{b} - \hat{D}^{(0)}{}_{\nu}\hat{A}^{(1)}{}_{\mu}{}^{a}{}_{b} + \mathcal{O}(\alpha') \ . \tag{63}$$

Note that this will imply that F has mixed tangent and normal indices at order $\sqrt{\alpha'}$. After some manipulation, Equation (59) becomes:

$$\hat{D}^{(0)}_{[\mu}F^{(0)}{}_{\nu\rho]kl}f^{(\perp)}{}_{a}{}^{kl} = 0 \ , \tag{64}$$

which is the Bianchi identity for F to $\mathcal{O}(\alpha'^0)$.

Since $f^{(\perp)}$ has entirely normal indices at $\mathcal{O}(\alpha'^0)$, we may write

$$f^{(\perp)}{}_{abc} = \mathcal{P}^{k}{}_{a}f^{(\perp)}{}_{kbc} \ . \tag{65}$$

Applying $\hat{D}$ to both sides, we find that

$$(1-\mathcal{P})^k{}_a \hat{D}_\rho f^{(\perp)}{}_{kbc} = \left(\hat{D}_\rho \mathcal{P}^k{}_a\right) f^{(\perp)}{}_{kbc} \ . \tag{66}$$

Combined with Equation (58), this means that:

$$\hat{D}_\mu e^a{}_\rho = -\alpha' \frac{3}{2} F^{(0)}{}_{\mu\rho kl} f^{(\perp)akl} \ . \tag{67}$$

We have found all of the constraints on the target space geometry and the left-moving supersymmetry current up to $\mathcal{O}(\alpha'^0)$. Let us summarize these results. The left-moving supersymmetry identifies a "tangent" subbundle of $\mathcal{V}$ with the tangent bundle of the target space; it also requires that the fibers of the orthogonal bundle are acted on by some Lie algebra in the adjoint representation. The one-form $e_\mu{}^a$ identifies the tangent subbundle of $\mathcal{V}$. The geometric structures of the target space and the tangent part of $\mathcal{V}$ are identified by the vanishing of (41)

$$R_{(+)\mu\nu\lambda\rho} = F_{\mu\nu ab} e_\lambda{}^a e_\rho{}^b \ , \tag{68}$$

and by the identification of the tangent part of f with the torsion in Equation (50). Note that with this identification the tangent part of G_- is the left-moving analog of the supersymmetry current in Equation (27). Equation (67) constrains the mapping $e_\mu{}^a$. Note that if the right hand side of Equation (67) is proportional to gauge and Lorentz covariant terms, then by dimensional analysis these terms must scale at least as $\sqrt{\alpha'}$, which is the only length scale available. The structure on the fibers of the normal subbundle of $\mathcal{V}$ is encoded in $f^{(\perp)}$. Equations (51) and (60) force $f^{(\perp)}$ to be, fiber by fiber, structure constants of some Lie group. The rotation of this group structure as we move around the target space is constrained by Equation (57); this equation can be derived from Equation (67). The remaining constraints consist of $\mathcal{O}(\sqrt{\alpha'})$ modifications of $\hat{A}$ and f; these modifications arise from nontrivial transverse gauge fields. The above constraints in the presence of transverse gauge curvature are the major new results of this section.

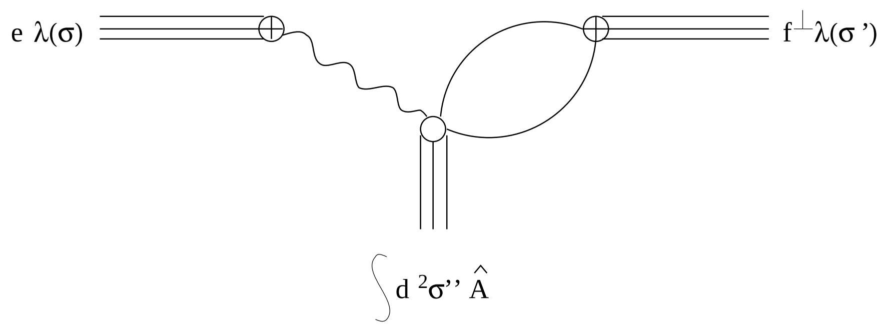


Figure 3

A one-loop $\mathcal{O}(\alpha'^{1/2})$ contribution to the commutator $\{G_-, G_-\}$. The wavy line denotes a boson propagator.

Since the constraints we have found require adding order $\mathcal{O}(\sqrt{\alpha'})$ terms to the action and to the supersymmetry current, the next step would be to check closure of the supersymmetry algebra at order $\mathcal{O}(\sqrt{\alpha'})$. We will not pursue this very far; however, to reassure ourselves that our solutions make sense at higher order, let us discuss one-loop corrections to the terms in Equation (48) which are quadratic in λ. The correction to $\hat{A}$ will show up in the first line of Equation (48) as an order $\mathcal{O}(\sqrt{\alpha'})$ piece of T_{--}, but this is due to the addition of $\hat{A}^{(1)}$ to the connection. In the second line $f^{(1)}$ and the order $\sqrt{\alpha'}$ corrections to $\hat{D}_{(-)}e$ combine to form the order $\sqrt{\alpha'}$ term in the second line of Equation (48):

$$\alpha' \frac{3}{2} e^{\mu}{}_{a} F^{(0)}{}_{\mu\rho kl} f^{(\perp)}{}_{b}{}^{kl} \partial_{-}\phi^{\rho} \lambda^{a} \lambda^{b} \ . \tag{69}$$

Closure requires that this be cancelled by one loop contributions to (48) at the appropriate order: such contributions will come from the commutator of $e\partial_{-}\phi\lambda$ with $f\lambda\lambda\lambda$. The diagram leading to the one-loop term proportional to

$$\alpha' e^{\mu}{}_{a} \partial_{\rho} \hat{A}_{\mu kl} f_{b}{}^{kl} \partial_{-}\phi^{\rho} \lambda^{a} \lambda^{b} \tag{70}$$

is shown in Figure 3. If we evaluate this diagram using the conventions stated in Appendix C, we find that the coefficient is 3/2. Note that the boson propagator in this diagram corresponds to the Green's function

$$\langle \partial_{+}\phi(\sigma) \partial_{-}\phi(\sigma') \rangle = 2\pi i \delta(\sigma - \sigma') \ . \tag{71}$$

Since the normal coordinate expansion is manifestly gauge and coordinate covariant, all terms in the commutator at this order contain $\hat{A}$ either in covariant derivatives or in curvature tensors (although for higher orders we should keep in mind the σ-model anomaly). Thus, so long as we know the correct coefficient in front of the term (70) we know the coefficient multiplying the term proportional to gauge curvature, of which (70) is a piece. In this case, then, the one loop order $\mathcal{O}(\sqrt{\alpha'})$ contribution to Equation (48) proportional to the gauge curvature is:

$$\alpha' \frac{3}{2} e^{\mu}{}_{a} F^{(0)}{}_{\rho\mu kl} f^{(\perp)}{}_{b}{}^{kl} \partial_{-}\phi^{\rho} \lambda^{a} \lambda^{b} \ , \tag{72}$$

which cancels the contribution from $f^{(1)}$ and $\hat{D}e$. We will leave the calculations of further one- and higher-loop contributions to the current commutators for future work.

We would like to note that although the $1/\sqrt{\alpha'}$ scaling complicates the α' expansion of the commutators it does not invalidate this expansion. The only negative powers of $\sqrt{\alpha'}$ arise from the currents themselves. σ-model counterterms will always be multiplied by positive powers of α' with respect to the bare Lagrangian; the divergences of the Green's functions may scale with large negative powers of α' if there are enough supersymmetry current insertions, but this is due to the scaling of external sources. One- and higher-loop divergences will always be higher order than the tree-level Green's function and will be removed by counterterms scaling with positive powers of α'.

5 Imposing the $U(1)_L$ symmetry

The restrictions on the geometry of the $(2,1)$ σ-model due to the $U(1)$ symmetry are derived in the same way as in the previous section. We require that the action be invariant; that the $U(1)$ current algebra contain no central term; and that the current be chiral:

$$\partial_+ J_- = 0 \ . \tag{73}$$

The conditions arising from invariance of the action and from chirality have been derived for the bosonic σ-model with torsion by Hull and Spence (1989, 1991), Jack *et.al.* (1990), and Rǒcek and Verlinde (1992); the generalization to (p,q) supersymmetry was given by Hull, Papadopolous and Spence (1991); the case of $N=2$ supersymmetry in superspace was discussed by Rǒcek and Verlinde (1992); and the gauging of heterotic σ-models was discussed by Hull (1994). We will rederive these results, and include the effect of quantum corrections arising at order $\mathcal{O}(\alpha'^0)$. J_- should be a gauge current and not a complex structure leading to $(2,2)$ supersymmetry (we will discuss the latter possibility in the next section), so we will demand that J_- be the top component of a supermultiplet, and we will find its dimension $(0,1/2)$ superpartner j_-.

We begin by examining the variation of the action due to the transformations given in equations (35)-(37). The bosonic terms in the variation of the action are:

$$\begin{aligned}\delta S_{\rm bos} = \int d^2\sigma \, \{ & \partial_+\epsilon \left[v^\mu \left(g_{\mu\nu} + b_{\mu\nu}\right) \partial_-\phi^\nu \right] \\ & + \partial_-\epsilon \left[v^\mu \left(g_{\mu\nu} - b_{\mu\nu}\right) \partial_+\phi^\nu \right] \\ & + \epsilon \left[\partial_+\phi^{(\mu} \partial_-\phi^{\nu)} 2\nabla_\mu v_\nu + \partial_+\phi^{[\mu} \partial_-\phi^{\nu]} \left(\mathcal{L}_v b_{\mu\nu}\right) \right] \Big\} \ ,\end{aligned} \tag{74}$$

where ∇ is the torsionless covariant derivative and $\mathcal{L}_v$ is the Lie derivative with respect to v. The vanishing of the first term of the last line of (74) means that v must be a Killing vector. The second term in the last line of (74) should combine with the first and second lines in a way that leaves only $\partial_+ J_-$. If we let

$$\mathcal{L}_v b = -d\omega \ , \tag{75}$$

then (74) becomes:

$$\begin{aligned}\delta S_{\rm bos} = \int d^2\sigma \, \Big\{ & \partial_+\epsilon \left[v^\mu \left(g_{\mu\nu} + b_{\mu\nu}\right) \partial_-\phi^\nu - \frac{1}{2}\omega_\mu \partial_-\phi^\mu \right] \\ & + \partial_-\epsilon \left[v^\mu \left(g_{\mu\nu} - b_{\mu\nu}\right) \partial_+\phi^\nu + \frac{1}{2}\omega_\mu \partial_+\phi^\mu \right] \Big\} \ .\end{aligned} \tag{76}$$

We wish the second line to vanish, so

$$\omega_\mu = -2 v^\rho \left(g_{\rho\mu} - b_{\rho\mu}\right) \ . \tag{77}$$

The first term in (76) is then a piece of $\partial_+\epsilon J_-$, as expected by Noether's theorem. Since v is a Killing vector, Equation (75) can be used to show that:

$$\nabla^{(-)}_\lambda v^\rho = 0 \ . \tag{78}$$

Combining this equation and Equation (16) we find that:

$$\mathcal{L}_v J^\mu{}_\nu = 0 \ . \tag{79}$$

Equation (75) also means that

$$\mathcal{L}_v H = 0 \tag{80}$$

to order $\mathcal{O}(\alpha'^0)$.

The only term term in δS quadratic in fermions is:

$$\begin{aligned}\delta S_{\lambda\lambda} = \int d^2\sigma \, \{ \, &\partial_+\epsilon\Sigma_{ab}\lambda^a\lambda^b \\ &+\epsilon\left[\frac{i}{2}v^\rho F_{\rho\lambda ab} + \hat{D}_\lambda\Sigma_{ab}\right]\partial_-\phi^\lambda\lambda^a\lambda^b\right\} \ .\end{aligned} \tag{81}$$

The first term is what we expect from Noether's theorem. The vanishing of the second term will be discussed below. The final term in the variation of the action is quartic:

$$\delta S_{\psi\psi\lambda\lambda} = \int d^2\sigma \left[\frac{1}{2}v^\rho \hat{D}^{(+)}{}_\rho F_{\mu\nu ab} - 2iF_{\mu\nu cb}\Sigma^c{}_a\right]\psi^\mu\psi^\nu\lambda^a\lambda^b \ . \tag{82}$$

Again, we will discuss the vanishing of this term below.

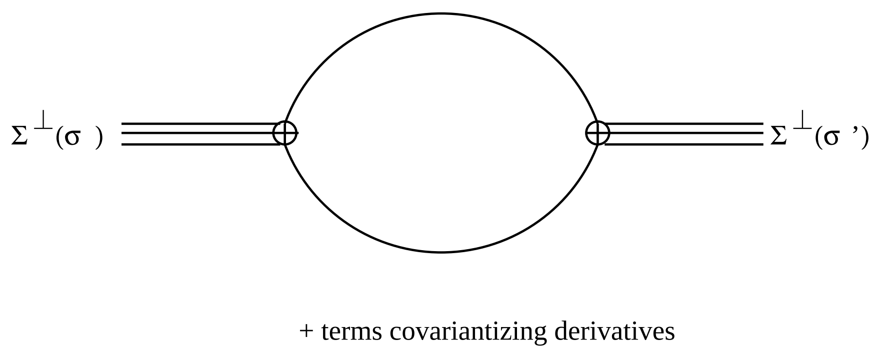


Figure 4

The one-loop $\mathcal{O}(\alpha'^0)$ contribution to $\{J_-, J_-\}$.

Next we wish to impose the vanishing of the commutator of the $U(1)$ current with itself. The classical bracket gives us:

$$\{J_-(\sigma), J_-(\sigma')\} = -2v^\mu v_\mu\delta'(\sigma-\sigma') - 2g_{\alpha\beta}v^\alpha\nabla_\lambda v^\beta\partial_-\phi^\lambda\delta(\sigma-\sigma') \ . \tag{83}$$

The antisymmetry of H allows us to turn ∇ into $\nabla_{(-)}$ in the second term, which vanishes. If the $U(1)$ is realized entirely as a spacetime isometry then this isometry must be null. However, if we have a piece of Σ which scales as $1/\sqrt{\alpha'}$, then at order $\mathcal{O}(\alpha'^0)$ we get a one-loop correction to the above which comes from the commutator of $\Sigma\lambda\lambda$ with itself. We compute this using the same strategy (and keeping in mind the same caveats) as the previous section. The relevant diagram is shown in Figure 4. Converting this term into a commutator expression gives us the expression for the order $\mathcal{O}(\alpha'^0)$ part of the commutator:

$$\begin{aligned}\{J_-(\sigma), J_-(\sigma')\} &= -2\left(v^\rho v_\rho + \alpha'\Sigma^{ab}\Sigma_{ab}\right)\delta'(\sigma-\sigma') \\ &\quad -\alpha'\Sigma^{ab}\hat{D}_\lambda\Sigma_{ab}\partial_-\phi^\lambda\delta(\sigma-\sigma')\ .\end{aligned} \tag{84}$$

to $\mathcal{O}(\alpha'^0)$ the final term vanishes, so that

$$v^\rho v_\rho + \alpha'\Sigma^{ab}\Sigma_{ab} = 0 \tag{85}$$

if the $U(1)$ current is to be anomaly-free.

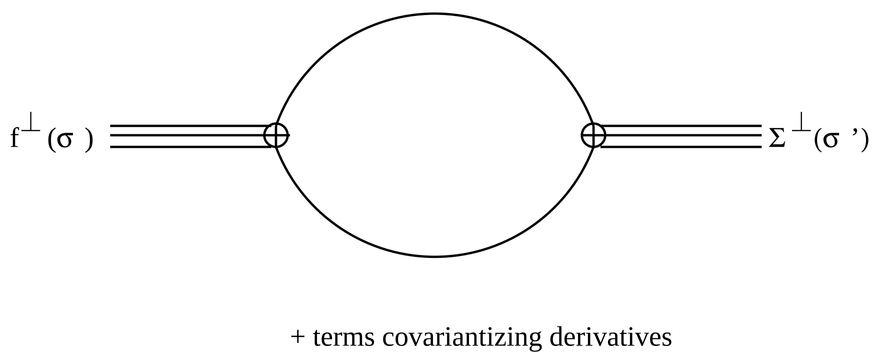


Figure 5
The one-loop $\mathcal{O}(\alpha'^0)$ contribution to $\{G_-, J_-\}$.

Supersymmetry requires that the Poisson bracket of the supersymmetry current with the top component ϕ_1 of a superfield is:

$$\{G_-(\sigma), \phi(\sigma')_1\} = -2h\phi_0\delta'(\sigma-\sigma') - \partial_-\phi_0\delta(\sigma-\sigma') \tag{86}$$

where h is the left conformal dimension of ϕ_1 and ϕ_0 is its superpartner, with left conformal dimension $h-1/2$. The classical Dirac bracket of the left-moving supersymmetry current with the left-moving $U(1)$ current is:

$$\begin{aligned}\{G_-(\sigma), J_-(\sigma')\}_{\text{tree}} &= -2\left(v^\mu e_{\mu a}\lambda^a\right)(\sigma')\delta'(\sigma-\sigma') + \\ &\quad \left[\left(2i\Sigma^k{}_a e_{\rho k} + H_{\rho\gamma\mu}e^\gamma{}_a v^\mu\right)\partial_-\phi^\rho\lambda^a - \partial_-\left(v^\mu e_{\mu a}\lambda^a\right) - \right. \\ &\quad \left.\left(iv^\mu\hat{D}_\mu f_{abc} + 6f_{kab}\Sigma^k{}_c\right)\lambda^a\lambda^b\lambda^c\right](\sigma')\delta(\sigma-\sigma')\ .\end{aligned} \tag{87}$$

In addition, the one loop part of the commutator of $\Sigma^{(\perp)}\lambda\lambda$ (where $\Sigma^{(\perp)}$ is the piece of Σ scaling as $1/\sqrt{\alpha'}$) and $f^{(\perp)}\lambda\lambda\lambda$ will contribute an order $\mathcal{O}(\alpha'^0)$ term. This term will come from the diagram shown in Figure 5. Using the fact that $\hat{D}\Sigma$ and $\hat{D}f^{(\perp)}$ are order $\mathcal{O}(\alpha'^0)$ or smaller, to this order the one contribution at order $\mathcal{O}(\alpha'^0)$ is:

$$\left\{-if^{(\perp)}{}_{abc}\lambda^a\lambda^b\lambda^c, \Sigma^{(\perp)}{}_{kl}\lambda^k\lambda^l\right\}_{1\text{loop}} = -6i\alpha' f^{(\perp)}_{abc}\Sigma^{(\perp)bc}\lambda^a(\sigma')\delta'(\sigma-\sigma')$$
$$-3i\alpha' f^{(\perp)}_{abc}\Sigma^{(\perp)bc}\left(\partial_-\lambda^a + \hat{A}_\mu{}^a{}_c\partial_-\phi^\mu\lambda^c\right)(\sigma')\delta(\sigma-\sigma') \ . \tag{88}$$

Using the fact that $\hat{D}f$ and $\hat{D}\Sigma$ vanish to this order, one can see that the last line is equal to:

$$-3i\alpha'\partial_-\left(f^{(\perp)}_{abc}\Sigma^{(\perp)bc}\lambda^a\right) \ , \tag{89}$$

as required by supersymmetry. Thus, if

$$e_{\rho k}\Sigma^k{}_a = \frac{i}{2}H_{\rho\alpha\mu}e^\alpha{}_a v^\mu \ , \tag{90}$$

and if the last line of Equation (87) vanishes, then J_- is the top component of a superfield and its dimension $(0,1/2)$ superpartner is:

$$j_- = \left(v^\mu e_{\mu a} + 3i\alpha' f^{(\perp)}{}_{abc}\Sigma^{(\perp)bc}\right)\lambda^a \ . \tag{91}$$

We also want to check that the anticommutator of j with itself vanishes. At order $\mathcal{O}(\alpha'^0)$ the tree level commutator will suffice. The condition for the vanishing of this anticommutator is:

$$v^\mu v_\mu - 9f_{akl}f^{amn}\Sigma^{kl}\Sigma_{mn} = 0 \ . \tag{92}$$

If we write

$$\Sigma_{ab} = 3iw^c f_{cab} \ , \tag{93}$$

then Equation (92) follows from the nilpotency of J_-.

Let us summarize what we have found so far:

$$\mathcal{L}_v g_{\mu\nu} = 0 \tag{94}$$
$$\mathcal{L}_v H_{\mu\nu\rho} = 0 \tag{95}$$
$$\nabla_{(-)\lambda}v^\rho = 0 \tag{96}$$
$$\hat{D}_\lambda\Sigma_{ab} = -\frac{i}{2}v^\rho F_{\rho\lambda ab} \tag{97}$$
$$\frac{1}{2}v^\rho\hat{D}_{(+)\rho}F_{\mu\nu ab} = 2i\Sigma^k{}_a F_{\mu\nu kb} \tag{98}$$
$$(1-\mathcal{P})^k{}_a\Sigma_{kb} = \frac{i}{2}e^\alpha{}_a e^\beta{}_b v^\rho H_{\alpha\beta\rho} \tag{99}$$
$$v^\rho\hat{D}_\rho f_{abc} = 6i\Sigma_{k[a}f^k{}_{bc]} \tag{100}$$
$$v^\rho v_\rho + \alpha'\Sigma^{ab}\Sigma_{ab} = 0 \ . \tag{101}$$

Not all of these equations are independent. The tangent part of Equation (97) can be shown with some work to follow from Equations (99) and (96). The tangent part of Equation (100) can be shown to be equivalent to Equation (80). The tangent part of Equation (98) is equivalent to:

$$\mathcal{L}_v R_{(+)\mu\nu\alpha\beta} = 0 \ . \tag{102}$$

The normal parts of Equations (98) and (100) take a a little more thought. f and $\hat{A}$ have order $\mathcal{O}(\sqrt{\alpha'})$ pieces, and Σ may as well. These will mix with order $\mathcal{O}(1/\sqrt{\alpha'})$ pieces of the currents to produce terms of $\mathcal{O}(\alpha'^0)$.

Let us break up Σ into an order $1/\sqrt{\alpha'}$ piece $\Sigma^{(\perp)}$, an order α'^0 piece $\Sigma^{(0)}$, and an order $\sqrt{\alpha'}$ piece $\Sigma^{(1)}$. $\Sigma^{(\perp)}$ and $\Sigma^{(1)}$ break up naturally into normal and tangent pieces; Equation (99) forces $\Sigma^{(\perp)}$ to be purely normal and fixes the tangent part of $\Sigma^{(0)}$. Now, one may use the vanishing of either Equation (56) or the second line of Equation (42)), to rewrite Equation (100) as:

$$\lambda^a \lambda^b \lambda^c \left(\frac{1}{2} e^{\alpha}{}_a v^\rho F_{\alpha\rho bc} - 6i \Sigma^k{}_a f_{kbc} \right) = 0 \ . \tag{103}$$

$(\Sigma^{(\perp)} + \Sigma^{(0)k}{}_a) f^{(\perp)}{}_{kbc}$ has all normal indices and so cannot cancel any piece of the first term in Equation (103); thus

$$\Sigma^{(\perp)}{}_{k[a} f^{(\perp)k}{}_{bc]} = \Sigma^{(0)}{}_{k[a} f^{(\perp)k}{}_{bc]} = 0 \ , \tag{104}$$

where each equation follows from a different order of $\sqrt{\alpha'}$ in Equation (103). If $\mathcal{P}^a{}_k \Sigma^k{}_b$ are written as in Equation (93), then Equation (104) follows from the Jacobi identity. The $\mathcal{O}(\alpha'^0)$ piece $\Sigma^{(\perp)k}{}_a f^{(1)}{}_{kbc}$ has two tangent indices and one normal index and cannot cancel off any piece of the first term of Equation (103). Using Equation (61), we find that

$$F_{\alpha\beta kl} f^{(\perp)akl} \Sigma^{(\perp)}{}_{ab} = 0 \ . \tag{105}$$

If we let $\Sigma^{(\perp)}_{ab} = f^{(\perp)}_{abc} w^c$ and $F_{\mu\nu ab} = F^{(c)}_{\mu\nu} f^{(\perp)}{}_{abc}$, then this equation implies that

$$F_{\mu\nu}{}^a{}_b w^b = \left[\hat{D}_\mu, \hat{D}_\nu \right]^a{}_b w^b = 0 \ . \tag{106}$$

The last line follows at this order from the fact that $\hat{D}\Sigma^{(\perp)}$ and $\hat{D}f$ also vanishes to lowest order; corrections will come at order $\mathcal{O}(\sqrt{\alpha'})$. The final $\mathcal{O}(\alpha'^0)$ piece of Equation (103) comes from $\Sigma^{(1)k}{}_a f^{(\perp)}{}_{kbc}$; thus Equation (103) requires that:

$$\Sigma^{(1)c}{}_a = \frac{3i}{4} \alpha' e^{\alpha}{}_a v^\beta F_{\alpha\beta kl} f^{(\perp)ckl} \ . \tag{107}$$

Note that higher order corrections to J_- and to G_- will also lead to higher order corrections to j_-.

The right hand side of Equation (98) has an $\mathcal{O}(1/\sqrt{\alpha'})$ piece $\Sigma^{(\perp)k}{}_a F_{\mu\nu kb}$, and two order α'^0 pieces, one from $\Sigma^{(0)}F$ and one from $\Sigma^{(\perp)}F^{(1)}$, where $F^{(1)}$ is the order $\mathcal{O}(\sqrt{\alpha'})$ piece of F arising from Equation (62). This last piece has one normal and one tangent gauge index, while the first piece and the left-hand side of Equation (98) have two normal indices and so vanish separately. The vanishing of the term with mixed indices,

$$\Sigma^k{}_a \left(\hat{D}_\mu \hat{A}^{(1)}{}_\nu{}^l{}_b G_{kl} - \hat{D}_\nu \hat{A}^{(1)}{}_\mu{}^l{}_b G_{kl} \right) \ , \tag{108}$$

follows from Equations (62) and (105) since the covariant derivatives of Σ, $\mathcal{P}$, and $f^{(\perp)}$ will show up at order $\sqrt{\alpha'}$. Using Equation (97) and the fact that

$$\left[\hat{D}_\mu, \hat{D}_\nu\right]^a{}_k \Sigma^k{}_b = F_{\mu\nu}{}^a{}_k \Sigma^k{}_b - F_{\mu\nu}{}^l{}_b \Sigma^a{}_l \ , \tag{109}$$

the rest of Equation (98) can be reduced to the $\mathcal{O}(\alpha'^0)$ part of the Bianchi identity:

$$\hat{D}^{(0)}_{[\rho} F^{(\perp)(0)a}_{\mu\nu]}{}_b = 0 \ . \tag{110}$$

In summary, the *independent* constraints arising from the $U(1)$ symmetry are that the worldvolume geometry admit an isometry (Equations (94) and (80)) which is covariantly conserved (Equation (96)). The part of the $U(1)$ current which is quadratic in λ must satisfy Equations (97) and (99). The isometry and this quadratic term are related by the requirement that the anomaly vanish, Equation (85)

Note that we could have started with the most general dimension $(0, 1/2)$ operator j_- and then found its dimension $(0,1)$ superpartner J_-. If we define v^a so that

$$(1-\mathcal{P})^a{}_b v^b = e^a{}_\mu v^\mu \tag{111}$$

$$\mathcal{P}^a{}_b v^b = w^b \tag{112}$$

then j_- will have the simple form

$$j_- = v_a \lambda^a \ . \tag{113}$$

However, by starting with the most general dimension $(0,1)$ current many of the results of this section can be used to find constraints on the geometry necessary for global $(2,2)$ and $(2,4)$ supersymmetry. (Recall that since the right-moving sector has $\hat{c} = 4$ and $N = 2$ supersymmetry, it automatically has global $(4,1)$ supersymmetry as well: see Eguchi *et.al.* (1989)). We will not derive these constraints here, but let us outline the necessary calculations. For $(2,2)$ supersymmetry one would start by demanding that the first line of Equation (87) vanish so that J_- was the bottom component of a superfield; the top component would be the additional supersymmetry charge $G_-^{(2)}$. Note that in this calculation Σ

would scale as α'^0. The conditions for invariance of the action have already been worked out; in order to find the rest of the constraints one would need to ensure closure of the $N = 2$ supersymmetry algebra (including the condition that J_- defines a $U(1)$ current algebra with level $c/3$). For $(2, 4)$ supersymmetry one would need to find two other dimension $(0, 1)$ operators in order to make up an $SU(2)$ current algebra; their dimension $(0, 3/2)$ superpartners would be the remaining supersymmetry currents of the $N = 4$ algebra.

6 Physical and geometric interpretation

The interpretation of the $N = (2, 1)$ theory as a mapping of a $2+2$-dimensional worldvolume into some spacetime is not obvious. In particular, we should not directly identify the vector bundle $\mathcal{V}$ with the spacetime or its tangent space. We can see this by thinking about examples with target space supersymmetry. In these cases the fermions living in the normal part of $\mathcal{V}$ are grouped into 8 groups of $SU(2)$ triplets and so $\mathcal{V}$ is broken up into 3 dimensional subspaces. In each of these subspaces two of the fermions are bosonized; the boson couples to a background $U(1)$ field which we will argue is related to a coordinate of the worldvolume in spacetime, while the fermion is its partner under worldsheet supersymmetry. We should not identify the chiral boson directly with a spacetime coordinate, just as one does not give the internal sector of the usual $(1, 0)$ heterotic string a spacetime interpretation; the chiral boson lives on a circle with a fixed radius and there is no graviton operator in the physical spectrum that would change this radius.

Instead, the spacetime should be directly related to the configuration space of the target space fields, in analogy to the soliton string constructions of Harvey and Strominger (1995) and Sen (1995); in these constructions the soliton string is the fundamental string of the dual theory, and the dual spacetime is the moduli space of zero-mode fluctuations around the soliton solution. In the $N = (2, 1)$ constructions the translation of this statement is that the spacetime of the target space string or membrane is parameterized for small fluctuations by the expectation value of the massless vertex operators of the $(2, 1)$ string (Kutasov and Martinec 1996; Kutasov, Martinec and O'Loughlin 1996).

In these $(2, 1)$ constructions, the vertex operators corresponding to string or membrane excitations are those for the Yang-Mills fields and for worldvolume gravity. We can see how the gauge field excitations are realized as coordinate excitations by writing

$$\hat{A}_\mu = J^\lambda{}_\mu h^{-1} \partial_\lambda h \ . \tag{114}$$

Here

$$h = e^{\phi^a t^a} \ , \tag{115}$$

where ϕ^a are scalars living in the adjoint of the gauge group and t are Hermitian generators of the Lie algebra, in the adjoint representation. Equation

(114) solves the constraint (17) (one may show this using Equation (14)); in complex coordinates (114) means that F_{ij} and $F_{\bar{i}\bar{j}}$ vanish. The vertex operators constructed from the currents of the internal gauge group in fact represent fluctuations of ϕ^a (Ooguri and Vafa 1991b). It seems natural to identify ϕ^a with the transverse coordinates of the 4-volume; this would mean that the spacetime is a group manifold. In the case $G = U(1)^8$, the group required for target space supersymmetry, the gauge fields will be parameterized by 8 scalars ϕ^a; the gauge field strength will be

$$F^a{}_{\mu\nu} = 2\partial_{[\mu} J^\alpha{}_{\nu]}\partial_\alpha\phi^a - 2J^\alpha{}_{[\mu}\partial_{\nu]}\partial_\alpha\phi^a \ . \tag{116}$$

In complex coordinates, this reads:

$$F^a{}_{i\bar{j}} = -2i\partial_i\partial_{\bar{j}}\phi^a \ . \tag{117}$$

The left-moving supersymmetry imposes a nontrivial structure on $\mathcal{V}$; in the action (7) there is no relation between geometric structures in $\mathcal{V}$ and the intrinsic geometry of the target space. This relation is encoded in the left-moving supersymmetry current (29), and in particular in the form $e_\mu{}^a$ which maps the tangent bundle of the target space into $\mathcal{V}$. Equations (67) and (68) show that the geometric structures of the target space and of the tangent part of $\mathcal{V}$ are related as well. Note that Equation (67) can be rewritten as:

$$\mathcal{P}^a{}_k\hat{D}_\rho\mathcal{P}^k{}_b = \frac{3}{2}\alpha' e^\mu{}_b F_{\rho\mu kl} f^{(\perp)akl} \ . \tag{118}$$

Although the details are not clear, these equations should be related to equations describing imbedded surfaces, such as the Gauss and Codazzi equations.[4] The first equation relates the intrinsic worldvolume curvature to the tangent part of F, and is reminiscent of the Gauss equation (the "Theorem Egregium"). The second equation relates $\hat{D}\mathcal{P}$ to $F^{(\perp)}$; $F^{(\perp)}$ is described by second derivatives of ϕ and could be related to the second fundamental form of the surface. There are many issues which need to be resolved before we can construct a definitive interpretation of the $(2,1)$ target space theory. One problem is that the normal and tangent gauge fields do not seem to have the same status. The transverse gauge fields have a natural interpretation as coordinates; the tangent gauge fields, however, seem to be mapped into the spin connection of the 4-volume. For example, if the gauge fields of the normal part of $\mathcal{V}$ are valued in the Lie algebra of $U(1)^8$, the tangent gauge fields have some non-Abelian structure induced from the local Lorentz group acting on tangent frames of the target space; it seems that the tangent gauge fields should not be related to coordinates in the same way as the normal gauge fields. Perhaps the $(2,1)$ σ model gives some sort of static gauge description of imbedded 4-volumes.

Supersymmetry seems to give us geometric constraints on the imbedding of 4-volumes into spacetime; the classical equations of motion for the target

[4]This interpretation was suggested by E. Martinec.

worldvolume are simply be the β-function equations for the heterotic string (Callan *et.al.* 1985; Fradkin and Tseytlin 1985b; Lovelace 1986; Sen 1986; Hull 1986a; Bonneau and Valent 1994). These equations seem to describe some sort of coupling of the self-dual Yang-Mills equations to self-dual gravity (Ooguri and Vafa 1991a,b). To see how these equations might be related to the equations of motion for the 4-volume, let us turn off gravity. The self-duality condition on F is:

$$\epsilon^{\mu\nu\lambda\rho} J_{\mu\nu} F_{\lambda\rho}{}^{a}{}_{b} = 0 \ . \tag{119}$$

If we use the ansätz (114) and work in complex coordinates, then these equations become (Nair and Schiff 1990,1992; Ooguri and Vafa 1991b):

$$g^{i\bar{j}} \partial_i \left(h^{-2} \partial_{\bar{j}} h^2 \right) = 0 \ . \tag{120}$$

In the Abelian theory this equation reduces to:

$$\left(\partial_1 \partial_{\bar{1}} - \partial_2 \partial_{\bar{2}} \right) \phi^a = 0 \ , \tag{121}$$

which is just the equation of motion for a free scalar field in $R^{(2,2)}$.

It is worth noting that one may phrase the classical equations of motion and constraints derived from the Nambu action for p-branes in terms of the Gauss and Codazzi equations and the vanishing of the trace of the second fundamental form of the imbedded worldvolume (Bandos *et.al.* 1995).

7 Conclusion

We have clearly taken only a small, if necessary, step towards understanding the classical and quantum dynamics of the target space of $N = (2, 1)$ strings. However, we can begin to see what questions to ask.

At present there is no obvious way to describe the imbedding of these world volumes in a more general, curved spacetime. One possibility comes from the fact that the self-dual Yang-Mills equations are equivalent to the classical equations of motion for a four dimensional generalization of the WZW model (Nair and Schiff 1990, 1992). This model looks like a 4-dimensional σ-model with an extra Wess-Zumino term, where the σ-model fields live in a group manifold. There is some belief that this theory exists as a quantum theory and has some current algebra structure similar to the 2-dimensional WZW model (Losev *et.al.* 1995). If we could understand better if and why this model exists, and how we might couple it to the gravity sector of the $N = (2, 1)$ target space, we might be able to generalize the WZW_4 model to more general target spaces.

We might also hope to make contact with other recent suggestions of $2 + 2$-dimensional worldvolumes in $10 + 2$-dimensional spacetimes. Many results have been obtained by compactifying F-theory on restricted classes of K3 surfaces (Vafa 1996), Calabi-Yau threefolds (Morrison and Vafa 1996), and other manifolds (Witten 1996a,b); one can make statements about the theory in curved

space, unlike the theory presented in this paper. On the other hand, it is not clear what the "fundamental" objects are; there are only some suggestions, discussed in the Introduction, that they might be four dimensional. Furthermore, the principles for constructing F-theory vacua are unknown; the aforementioned results for compactifications of F-theory come from comparing the moduli space of the compactification manifold with the moduli space of other theories. It would be nice if there was a direct connection between F-theory and the $(2,1)$ string, as there are some analogous structures, but how to make this link is not obvious.

It may be that if and when we successfully define a perturbation theory of sums over 4-volumes, that this theory will break down at high energies and large orders just as string theory seems to. This is no reason not to go forward; string theory has done remarkably well in providing hints of its underlying structure, and has even given us enough information (by constraining the low-energy vacua) to find some of its nonperturbative physics. Perhaps this theory of 4-volumes will do the same.

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A Target space and worldsheet geometry

A.1 Target space geometry

In general the target space of the $N=(2,1)$ string will have a metric $g_{\mu\nu}$, with signature $(2,2)$, and an antisymmetric 2-form $b_{\mu\nu}$ from which we may form the 3-form field strength

$$H_{\mu\nu\lambda} = \partial_\mu b_{\nu\lambda} + \partial_\nu b_{\lambda\mu} + \partial_\lambda b_{\mu\nu} \ , \tag{122}$$

or equivalently,

$$H = 3db \ . \tag{123}$$

H is a torsion tensor: we define the torsionful connection as:

$$\Gamma_{(\pm)}{}^{\mu}_{\nu\lambda} = \Gamma_{(c)}{}^{\mu}_{\nu\lambda} \pm \frac{1}{2} H^{\mu}_{\nu\lambda} \tag{124}$$

where $\Gamma_{(c)}$ is the Christoffel connection. Covariant derivatives with torsion are defined as usual:

$$\begin{aligned}\nabla_{(\pm)\lambda} K^{\mu\nu\ldots}{}_{\alpha\beta\ldots} \quad = \quad & \partial_\lambda K^{\mu\nu\ldots}{}_{\alpha\beta\ldots} + \Gamma_{(\pm)}{}^{\mu}_{\lambda\rho} K^{\rho\nu\ldots}{}_{\alpha\beta\ldots} + \Gamma_{(\pm)}{}^{\nu}_{\lambda\rho} K^{\mu\rho\ldots}{}_{\alpha\beta\ldots} + \ldots \\ & - \Gamma_{(\pm)}{}^{\rho}_{\lambda\alpha} K^{\mu\nu\ldots}{}_{\rho\beta\ldots} - \Gamma_{(\pm)}{}^{\rho}_{\lambda\beta} K^{\mu\nu\ldots}{}_{\alpha\rho\ldots} + \ldots \; . \end{aligned} \tag{125}$$

The curvature with torsion is defined via the equation

$$\left[\nabla_{(\pm)\mu}, \nabla_{(\pm)\nu}\right] v^\lambda = R_{(\pm)}{}^{\lambda}{}_{\rho\mu\nu} v^\rho \mp H^\alpha{}_{\mu\nu} \nabla_{(\pm)\alpha} v^\lambda , \tag{126}$$

and it can be written in terms of the connection as:

$$R_{(\pm)}{}^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \Gamma_{(\pm)}{}^{\rho}{}_{\nu\lambda} + \Gamma_{(\pm)}{}^{\rho}{}_{\mu\gamma} \Gamma_{(\pm)}{}^{\rho}{}_{\mu\gamma} - (\mu \longleftrightarrow \nu) \; . \tag{127}$$

$R_{(\pm)}$ satisfies the following identities:

$$R_{(\pm)\mu\nu\alpha\beta} \quad = \quad -R_{(\pm)\nu\mu\alpha\beta} = -R_{(\pm)\nu\mu\alpha\beta} = -R_{(\pm)\nu\mu\beta\alpha} \tag{128}$$

$$R_{(+)\mu\nu\alpha\beta} \quad = \quad R_{(-)\alpha\beta\mu\nu} \; . \tag{129}$$

The left-moving fermions live on a 28-dimensional vector bundle $\mathcal{V}$ with signature $(26, 2)$ and metric G_{ab}. Generally, in each fiber we split the space into a 4-dimensional space with signature $(2, 2)$ and an orthogonal Euclidean 24-dimensional space. This gives us two separate vector bundles which are complements in $\mathcal{V}$. The bundle of 4-dimensional vector spaces will be identified with the tangent bundle of the target space, while the other 24 will live in some internal vector bundle fibered over the target spacetime. If $A_\mu{}^a{}_b$ is an antisymmetric connection on this bundle, then the metric-compatible connection is:

$$\hat{A}_\mu{}^a{}_b = A_\mu{}^a_b + \frac{1}{2} G^{ac} \partial_\mu G_{cb} \; , \tag{130}$$

which is not antisymmetric. In this paper we use the covariant derivative:

$$\begin{aligned}\hat{D}_\lambda M^{ab\ldots}{}_{cd\ldots} = \partial_\lambda M^{ab\ldots}{}_{cd\ldots} + \hat{A}_\mu{}^a{}_k M^{kb\ldots}{}_{cd\ldots} + \\ \hat{A}_\mu{}^b{}_k M^{ak\ldots}{}_{cd\ldots} + \ldots - \hat{A}_\mu{}^k{}_c M^{ab\ldots}{}_{kd\ldots} - \hat{A}_\mu{}^k{}_d M^{ab\ldots}{}_{ck\ldots} - \ldots\end{aligned} \tag{131}$$

$\hat{D_\pm}$ is covariantized with respect to both tangent space and vector bundle indices in the obvious way. The field strength that appears in the action is

$$F_{\mu\nu}{}^a{}_b = \partial_\mu \hat{A}_\nu{}^a{}_b + \hat{A}_\mu{}^a{}_c \hat{A}_\nu{}^c{}_b - (\mu \longleftrightarrow \nu) \; . \tag{132}$$

We raise and lower indices directly on this field strength tensor rather than defining $F_{\mu\nu ab}$ as $\partial_\mu \hat{A}_{\nu ab}(+\ldots)$; with some work one can show that F_{ab} is antisymmetric in the gauge indices. Note that unlike the field strength defined by Hull and Witten (1985), here F is in fact the commutator of gauge covariant derivatives:

$$\left[\hat{D}_\mu, \hat{D}_\nu\right] v^a = F_{\mu\nu}{}^a{}_b v^b \; . \tag{133}$$

We may also work in the tangent space of $\mathcal{V}$ using the vielbein ρ_a^A, where

$$\eta_{AB}\rho_a^A\rho_b^B = G_{ab}\ ,\ G^{ab}\rho_a^A\rho_b^B = \eta^{AB} \tag{134}$$

Here lowercase indices $(a, b, ...)$ are vector bundle indices, raised and lowered with the curved metric G and uppercase indices are the indices for the tangent space to the bundle with the flat metric η. Tensors with uppercase indices can be converted to tensors with lowercase indices by contraction with ρ. The covariant derivatives are as in Equation (131), with the lowercase indices replaced by uppercase indices and the connection $\hat{A}$ replaced by the antisymmetric connection one-form

$$\omega_{\mu\, AB} = \hat{A}_{\mu ab}\rho_A^a\rho_B^b + \rho_{Ac}\partial_\mu\rho_B^c\ . \tag{135}$$

If we have a complex structure with a vanishing Nienhuis tensor we may choose a coordinate system such that:

$$J^i{}_{\bar{j}} = J^{\bar{i}}{}_j = 0 \tag{136}$$
$$J^i{}_j = i\delta^i{}_j \tag{137}$$
$$J^{\bar{i}}{}_{\bar{j}} = -i\delta^{\bar{i}}_{\bar{j}}\ . \tag{138}$$

In these coordinates the flat space metric in $2+2$ dimensions is:

$$ds^2 = -dz^1dz^{\bar{1}} + dz^2dz^{\bar{2}}\ . \tag{139}$$

A.2 Worldsheet (super-) geometry

Throughout this paper, we work in conformal gauge. The worldsheet metric is:

$$ds^2 = e^{2\phi}\left(-d\tau^2 + d\sigma^2\right)\ . \tag{140}$$

We will be working on the cylinder, so

$$-\infty < \tau < \infty,\ 0 \leq \sigma \leq 2\pi\ . \tag{141}$$

Light cone derivatives are defined as:

$$\partial_\pm = \frac{\partial}{\partial\tau} \pm \frac{\partial}{\partial\sigma}\ , \tag{142}$$

so that

$$\sigma_\pm = \frac{\tau \pm \sigma}{2}\ . \tag{143}$$

The $(1,0)$ superfields are:

$$\Phi^\mu \ = \ \phi^\mu + \theta_+\psi^\mu \tag{144}$$
$$\Lambda^a \ = \ \lambda^a + \theta_+ F^a \tag{145}$$

where F is an auxiliary field. The superspace derivative is:

$$D_+ = \frac{\partial}{\partial\theta_+} + i\theta_+\partial_+ \ . \tag{146}$$

For $(2,0)$ superspace we follow the conventions of Dine and Seiberg (1986). There are two commuting Grassman coordinates θ_+, θ_+^* paired with σ_+. The superspace derivatives are:

$$D_+ = \frac{\partial}{\partial\theta_+} + i\theta_+^*\partial_+ \tag{147}$$

$$D_+^* = \frac{\partial}{\partial\theta_+^*} + i\theta_+\partial_+ \tag{148}$$

$$\tag{149}$$

Chiral and antichiral superfields are written in complex coordinates with conjugate indices:

$$D_+^*\Phi^i = 0 \Longrightarrow \Phi^i = \phi^i + \sqrt{2}\theta_+\psi^i - i\theta_+^*\theta_+\partial_+\phi \tag{150}$$

$$D_+\Phi^{\bar{j}} = 0 \Longrightarrow \Phi^{\bar{j}} = \phi^{\bar{j}} + \sqrt{2}\theta_+^*\psi^{\bar{j}} + i\theta_+^*\theta_+\partial_+\phi^{\bar{j}} \ . \tag{151}$$

B Fundamental Poisson and Dirac brackets of the $N = (2,1)$ σ-model

First let us recall the construction of Poisson brackets for systems with fermions (for a clear explanation see ch. 6, sections 4 and 5 of Henneaux and Teitelboim (1992), from which the discussion in this paragraph was lifted; we repeat this discussion in order to explain our conventions). Let us look at a system with commuting phase space coordinates and momenta (q,p) and anticommuting coordinates and momenta (θ,π); the latter define an obvious $\mathcal{Z}_2$ grading, so we can define monomials and by extension certain functions as being even or odd with respect to this grading. We define partial derivatives as acting from the right; differentials are given by the formula

$$\delta F(z) = \delta z_i \frac{\partial F}{\partial z_i} \ . \tag{152}$$

Given a Lagrangian $L(q,\dot{q},\theta,\dot{\theta})$, the Hamiltonian as:

$$H = \dot{q}p + \dot{\theta}\pi - L \tag{153}$$

If we set to zero the infinitesimal variation of H with respect to time translations, we of course find Hamilton's equations:

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{154}$$

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p} && (155)\\
\dot{\pi} &= -\frac{\partial H}{\partial \theta} && (156)\\
\dot{\theta} &= -\frac{\partial H}{\partial \pi}\ . && (157)
\end{aligned}$$

Note the minus sign in Equation (157). Now if H is an even function then we can compute

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\} \tag{158}$$

where if B is an even function the Poisson bracket above is defined as

$$\{A, B\} = \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} + \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} + \frac{\partial B}{\partial \pi_a}\frac{\partial A}{\partial \theta^a} + \frac{\partial B}{\partial \theta^a}\frac{\partial A}{\partial \pi_a}\ . \tag{159}$$

If B is odd and A is even then we define formally

$$\{A, B\} \equiv -\{B, A\}\ , \tag{160}$$

where the bracket on the right hand side of this equation is defined above. For A and B both odd, we should be a little careful: if we want the bracket to be symmetric and to have the associative properties of an anticommutator

$$\{AB, C\} = A\{B, C\} - \{A, C\}B\ , \tag{161}$$

then the signs in front of the last two terms of Equation (159) are reversed. The final expression for the commutator of functions A and B with definite grading is (Henneaux and Teitelboim 1992):

$$\{A, B\} = \left(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} + \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}\right) + (-1)^{\sigma(A)}\left(\frac{\partial A}{\partial \theta^i}\frac{\partial B}{\partial \pi_i} + \frac{\partial A}{\partial \pi^i}\frac{\partial B}{\partial \theta_i}\right)\ . \tag{162}$$

where $\sigma(A)$ is $+1$ if A is even and -1 if A is odd The canonical Poisson brackets are:

$$\{q, p\} = 1\ ;\ \{\theta, \pi\} = -1\ . \tag{163}$$

If τ is chosen as the worldsheet time variable, the canonical momenta of the action (5)+(7) are:

$$\begin{aligned}
p_{\phi,\mu} &= g_{\mu\nu}\dot{\phi}^\nu - b_{\mu\nu}\phi'^\nu + \frac{i}{2}g_{\alpha\rho}\Gamma_{(+)}{}^\rho{}_{\mu\beta}\psi^\alpha\psi^\beta + \frac{i}{2}A_{\mu ab}\lambda^a\lambda^b && (164)\\
\pi_{\psi,\mu} &= -\frac{i}{2}g_{\mu\nu}\psi^\nu && (165)\\
\pi_{\lambda,a} &= = -\frac{i}{2}G_{ab}\lambda^b\ . && (166)
\end{aligned}$$

This system is clearly constrained. We follow Dirac's procedure (Dirac 1967; see also Hanson, Regge and Teitelboim 1976 for an introduction and many examples). The constraints

$$\chi_\mu = \pi_\mu + \frac{i}{2} g_{\mu\nu}\psi^\nu = 0 \tag{167}$$

$$\chi_a = \pi_a + \frac{i}{2} G_{ab}\lambda^a = 0 \tag{168}$$

are second class:

$$\{\chi_\mu, \chi_\nu\} \equiv C_{\mu\nu} = -i g_{\mu\nu} \tag{169}$$

$$\{\chi_a, \chi_b\} \equiv C_{ab} = -i G_{ab} \ . \tag{170}$$

We may compute the Dirac brackets in standard fashion to find:

$$\{p_{mu}(\sigma), \phi^\nu(\sigma')\}_D = \delta_\mu{}^\nu \delta(\sigma - \sigma') \tag{171}$$

$$\{\psi^\mu(\sigma), \psi^\nu(\sigma')\}_D = -i g^{\mu\nu}\delta(\sigma - \sigma') \tag{172}$$

$$\{p_\mu, \psi^\nu\}_D = \frac{1}{2} g^{\nu\beta} g_{\beta\rho,\mu}\psi^\rho \delta(\sigma - \sigma') \tag{173}$$

$$\{p_\mu, \lambda^a\}_D = \frac{1}{2} G^{ad} G_{db,\mu}\lambda^b \tag{174}$$

$$\{p_\mu, p_\nu\}_D = -\frac{i}{4} g^{\alpha\beta} g_{\alpha\lambda,\mu} g_{\beta\nu,\rho}\psi^\lambda\psi^\rho + -\frac{i}{4} G^{ab} G_{ac,\mu} g_{bd,\nu}\lambda^c\lambda^d \ . \tag{175}$$

The last three brackets are due to the metric factor in the kinetic term for ψ and λ. If we rotate the fermions with the appropriate vielbeins, so that the kinetic terms are the standard flat-space terms, the last three brackets above will vanish.

In calculations in the body of the paper, all brackets are Dirac brackets (so we drop the subscript), and we lift them to quantum commutators in the standard way.

C Conventions for worldsheet Green's functions

In this paper we are interested in the singular short-distance behavior of commutators and operator products. For these purposes we may assume that σ extends over the real line. If the action in flat space is normalized like so:

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \left(\eta_{\mu\nu}\partial_+\phi^\mu\partial_-\phi^\nu + i\eta_{ab}\lambda^a\partial_+\lambda^b + i\eta_{\mu\nu}\psi^\mu\partial_-\psi^\nu \right) \ , \tag{176}$$

then the ϕ propagator will be:

$$\langle \phi^\mu(\sigma)\phi^\nu(\sigma')\rangle = 2\pi i\alpha'\eta^{\mu\nu} \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik\cdot(\sigma-\sigma')}}{k^2 + i\epsilon} \tag{177}$$

$$= -\frac{\alpha'}{2}\eta^{\mu\nu} \ln(\sigma_- - \sigma'_-)(\sigma_+ - \sigma'_+) \ , \tag{178}$$

while the λ propagator will be:

$$\langle \lambda^a(\sigma_-)\lambda^b(\sigma'_-)\rangle = 2\pi i\alpha'\eta^{ab}\int \frac{d^2k}{(2\pi)^2}\frac{k_+ e^{ik\cdot(\sigma-\sigma')}}{k^2+i\epsilon} \tag{179}$$

$$= \frac{-i\alpha'\eta^{ab}}{2(\sigma_- - \sigma'_-)} \ . \tag{180}$$

and similarly for ψ, with $\sigma_-, k_+ \rightarrow \sigma_+, k_-$. If we wish to convert operator product singularities to equal-time commutator singularities, we may use the formula:

$$\frac{1}{2(\sigma_- - \sigma'_-)} \longrightarrow \delta(\sigma - \sigma') \ . \tag{181}$$

References

Ademollo M *et.al.* 1976a *Phys. Lett.* B **62** 105

Ademollo M *et.al.* 1976b *Nucl. Phys.* B **111** 77

Aldazabal G, Hussain F and Zhang R 1987 *Phys. Lett.* B **185** 89

Alvarez-Gaumé L and Freedman D Z 1981 *Commun. Math. Phys.* **80** 443

Alvarez-Gaumé L, Freedman D Z and Mukhi S 1981 *Ann. Phys, NY* **134** 85

Antoniadis I, Bachas C, Kounnas C and Windey P 1986 *Phys. Lett.* B **171** 51

Atick J J and Witten E 1988 *Nucl. Phys.* B **310** 291

Bandos I A, Sorokin D, Tonin M, Pasti P and Volkov D V 1995 *Nucl. Phys.*B **446** 79

Banks T and Dixon L J 1988 *Nucl. Phys.* B **307** 93

Banks T, Dixon L J, Friedan D and Martinec E 1988 *Nucl. Phys.* B **299** 613

Banks T, Nemeschansky D and Sen A 1986 *Nucl. Phys.* B **277** 67

Becker K, Becker M and Strominger S 1995 *Nucl. Phys.* B **456** 130

Blencowe M P and Duff M J 1988 *Nucl. Phys.* B **310** 387

Bluhm R, Dolan L and Goddard P 1987 *Nucl. Phys.* B **289** 364

Bonneau G and Valent G 1994 *Class. Quant. Grav.* **11** 1133

Braden H W 1987 *Nucl. Phys.* B **291** 516

Callan C G, Friedan D, Martinec E J and Perry M J 1985 *Nucl. Phys.* B **262** 593

Coleman S 1975 *Phys. Rev.* D **11** 2088

D'Adda A and Lizzi F 1987 *Phys. Lett.* B **191** 85

Delduc F, Kalitzin F and Sokatchev E 1990 *Class. Quant. Grav.* **7** 1567

Dine M and Seiberg N 1986 *Phys. Lett.* B **180** 364

Dirac P A M (1967) *Lectures on Quantum Mechanics* (New York: Academic Press Inc.)

Dixon L J, Kaplunovsky V S and Vafa C 1987 *Nucl. Phys.* B **294** 43

Eguchi T, Ooguri H, Taormina A and Yang S-K 1989 *Nucl. Phys.* B **193**.

Fradkin E S and Tseytlin A A 1981 *Phys. Lett.* B **106** 63

——– 1985a *Phys. Lett.* B **162** 295

——– 1985b *Nucl. Phys.* B **261** 1

Gates S J, Howe P S and Hull C M 1989 *Phys. Lett.* B **227** 49

Goddard P, Nahm W and Olive D 1985 *Phys. Lett.* B **160** 111

Goddard P, Kent A and Olive D 1986 *Commun. Math. Phys.* **103** 105

Goddard P and Olive D 1986 *Int. J. Mod. Phys.* A **1** 303

Green M B 1987 *Nucl. Phys.* B **293** 593

——— 1994 *Phys. Lett.* B **329** 435

Greene B P, Morrison D R and Strominger A 1995 *Nucl. Phys.* B **451** 109

Gross D J, Harvey J A, Martinec E and Rohm R 1985 *Nucl. Phys.* B **256** 253

——— 1986 *Nucl. Phys.* B **267** 75

Gross D J and Mende P F 1987 *Phys. Lett.* B **197** 129

——— 1988 *Nucl. Phys.* B **303** 407

Halpern M 1975 *Phys. Rev.* D **12** 1684

Hanson A, Regge T and Teitelboim C 1976 *Constrained Hamiltonian Systems* (Rome: Accademia Nazionale dei Lincei)

Harvey J A and Strominger A 1995 *Nucl. Phys.* B **449** 535; erratum *Nucl. Phys.* B **458** 456

Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)

Howe P S and Papadopoulos G 1988 *Class. Quant. Grav.* **5** 1647

Hull C M 1986a *Phys. Lett.* B **178** 357

——— 1986b *Nucl. Phys.* B **267** 266

——— 1994 *Mod. Phys. Lett.* A **9** 161

——— 1995 String dynamics at strong coupling *Preprint* QMW-95-50, hep-th/9512181

Hull C M, Papadopoulos G and Spence B 1991 *Nucl. Phys.* **363** 593

Hull C M and Spence B 1989 *Phys. Lett.* B **232** 204

——— 1990 *Nucl. Phys.* B **345** 493

——— 1991 *Nucl. Phys.* B **353** 379

Hull C M and Townsend P K 1986 *Phys. Lett.* B **178** 187

Hull C M and Witten E 1985 *Phys. Lett.* B **160** 398

Jack I, Jones D R T, Mohammedi N and Osborn H 1990 *Nucl. Phys.* B **332** 359

Kawai H, Lewellyn D C and Tye S-H H 1986a *Phys. Rev.* D **34** 3794

——— 1986b *Phys. Rev. Lett.* **57** 1832

——— 1987a *Nucl. Phys.* B **288** 1

——— 1987b *Phys. Lett.* B **191** 63

Kutasov D and Martinec E 1996 New Principles for String/Membrane Unification, *Preprint* EFI-96-04, hep-th/9602049

Kutasov D, Martinec E and O'Loughlin M 1996 Vacua of M-theory and N=2 Strings, *Preprint* EFI-96-07, hep-th/9603116

Losev A, Moore G, Nekrasov N and Shatashvili S 1995 Four-Dimensional Avatars of Two-Dimensional RCFT, *Talk presented at the USC Strings '95 conference*, *Preprint* PUPT-1564,ITEP-TH.5/95,YCTP-P15/95, hep-th/9509151

Lovelace C 1986 *Nucl. Phys.* B **273** 413

Mathur S and Mukhi S 1987 *Phys. Rev.* D **36** 465

——— 1988 *Nucl. Phys.* B **302** 130

Moore G and Nelson P 1984 *Phys. Rev. Lett.* **53** 1519

——— 1985 *Commun. Math. Phys.* **100** 83

Morrison D R and Vafa C 1996 Compactifications of F-Theory on Calabi-Yau Threefolds I *Preprint* DUKE-TH-96-106, HUTP-96-A007 hep-th/9602114; Compactifications of F-Theory on Calabi-Yau Threefolds II, *Preprint* DUKE-TH-96-107, HUTP-96-A012, hep-th/9603161

Nair V P and Schiff J 1990 *Phys. Lett.* B **246** 423

——— 1992 *Nucl. Phys.* B **371** 329

Ooguri H and Vafa C 1990 *Mod. Phys. Lett.* A **5** 1389

——— 1991a *Nucl. Phys.* B **361** 469

——— 1991b *Nucl. Phys.* B **367** 83

Pierce D M 1986 A (1,2) Heterotic String With Gauge Symmetry, *Preprint* IFP-604-UNC, hep-th/9601125

Polchinski J 1988 *Phys. Lett.* B **209**, 252

——— 1994 What is String Theory? *Lectures from the 1994 Les Houches summer school*, Fluctuating geometries in statistical mechanics and field theory, *Preprint* NSF-ITP-94-97, hep-th/9411028

Rŏcek M and Verlinde E 1992 *Nucl. Phys.* B **373** 630

Schwarz J A 1995a *Phys. Lett.* **360** 13; erratum *Phys. Lett.* B **364** 252

——— 1995b Superstring dualities *Preprint* CALT-68-2019, hep-th/9509148

——— 1995c *Phys. Lett.* B **367** 97

Schwimmer A and Seiberg N 1987 *Phys. Lett.* B **184** 191

Sen A 1985 *Phys. Rev. Lett.* **55** 1846, *Phys. Rev.* D **32** 2102

——— 1986 *Phys. Lett.* B **166** 300; *Phys. Lett.* B **174** 277; *Nucl. Phys.* B **278** 289

——— 1995 *Nucl. Phys.* B **450** 103

Shenker S H 1995 Another length scale in string theory? *Preprint* RU-95-53, hep-th/9509132

Townsend P K 1995 P-Brane Democracy *Preprint* DAMPT-95-34, hep-th/9507048

Tseytlin A A 1996 Self-duality of Born-Infeld action and Dirichlet 3-brane of type IIB superstring theory *Preprint* Imperial/TP/95-96/26, hep-th/9602064

Vafa C 1996 Evidence for F-Theory *Preprint* HUTP-96-A004, hep-th/9602022

van Nieuwenhuizen P 1987 *Santiago 1987, Proceedings, Quantum mechanics of fundamental systems* (New York: Plenum)

Windey P 1986 *Commun. Math. Phys.* **105** 511

Witten E 1988 *Phys. Rev. Lett.* **61** 670

——— 1996a Phase Transitions in M Theory and F Theory *Preprint* IASSNS-HEP-96-26, hep-th/9603150

——— 1996b Nonperturbative Superpotentials in String Theory *Preprint* IASSNS-HEP-96-29, hep-th/9604030